On the topology of locally volume collapsed Riemannian 3-orbifolds

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Abstract

We study the geometry and topology of Riemannian 3-orbifolds which are locally volume collapsed with respect to a curvature scale. We show that a sufficiently collapsed closed 3-orbifold without bad 2-suborbifolds either admits a metric of nonnegative sectional curvature or satisfies Thurston's Geometrization Conjecture. We also prove a version with boundary. Kleiner and Lott independently proved similar results [KL11].

Contents

1 Introduction

We study the geometry and topology of Riemannian 3-orbifolds which are locally volume collapsed with respect to a curvature scale. Such orbifolds are expected to occur as the *thin* part of solutions of the orbifold version of Perelman's Ricci flow with surgery (as constructed on manifolds in [Pe03]) after sufficiently long time. Our main result (Theorem 4.13) concerns the topology of locally collapsed 3-orbifolds. We show that a sufficiently collapsed closed 3-orbifold without bad 2-suborbifolds either admits a metric of nonnegative sectional curvature or satisfies Thurston's Geometrization Conjecture, i.e. has a connected sum decomposition (by spherical surgeries) into components which in turn admit a (toric) JSJ-decomosition into *geometric* components. Closed 3-orbifolds with nonnegative sectional curvature are also expected to be geometric by an orbifold version of Hamilton's corresponding result for 3-manifolds [Ha82].

In order to avoid the use of an orbifold version of Perelman's Stability Theorem we prove our result under an additional regularity assumption. We require uniform control on the derivatives of the curvature tensor up to some (sufficiently large) finite order.

We also prove a version with boundary of our result (Theorem 5.2). We expect that the assumptions are sufficiently general to apply to the thin part of a solution of the orbifold Ricci flow with surgery if the thick-thin decomposition is nontrivial (i.e. in a situation of partial collapse).

Corresponding results for collapsed orientable 3-manifolds have been stated without proof in [Pe03] and proved in [SY05], [MT08], [KL10] and [BBBMP10]. After writing this paper, we learned that Kleiner and Lott independently proved results similar to our main result, cf. [KL11, Prop. 9.7]. Their method is an extension of their work [KL10] in the manifold case to the orbifold case, whereas our approach is closer to an extension of the approach in [MT08].

The paper is organized as follows: In section 2, we first review basic facts on orbifolds in low dimensions. We then discuss decompositions of 3-orbifolds along spherical and toric 2-suborbifolds and prove that *graph* orbifolds in the sense of Waldhausen (cf. section 2.3.3) satisfy Thurston's Geometrization Conjecture (Corollary 2.9).

In the third section, we discuss a coarse stratification of roughly 2-dimensional Alexandrov spaces. More precisely, we use a conical approximation argument to show that the points in such a space which do not admit 1-strainers of a certain length and quality accumulate in isolated regions. Outside these regions, the Alexandrov space is 1-strained which allows us to perform a (coarse) dimension reduction by considering cross sections to these strainers. We further distinguish points according to whether they lie in coarse necks, edges or the interior of the Alexandrov space and study their geometric properties. These considerations are similar in spirit to considerations in [MT08] and [KL10].

In section 4, we restrict our attention to closed volume collapsed 3-orbifolds. We consider them as Alexandrov spaces which are roughly of dimension ≤ 2 and promote their coarse stratification to a certain decomposition into 3-suborbifolds. To determine the local topology of the components in this decomposition, we use a variation (and extension to additional situations) of the blow-up arguments in [SY00]. We derive a graph decomposition of the collapsed 3-orbifolds. Combined with the results of section 2, the main result follows.

The fifth section contains a generalization of Theorem 4.13 to compact orbifolds with boundary where we require that neighbourhoods of the boundary are close to pieces of hyperbolic cusps (cf. Theorem 5.2).

2   Decompositions of 3-orbifolds along 2-suborbifolds

2.1   Orbifolds

We refer to [BMP03, sec. 2] for a more detailed discussion of orbifolds.

2.1.1   Smooth orbifolds

Roughly speaking, an orbifold is a space which looks locally like the orbit space of a linear action by a finite (orthogonal) group. Important examples are the orbit spaces of properly discontinuous group actions on manifolds, in particular of finite group actions. These orbifolds are called *good*, respectively, *very good*. To exclude exotic local phenomena, e.g. when working in the topological category, one has to require the group actions to be locally linearizable; this is automatic in the smooth category.

More formally, an n-dimensional *smooth orbifold O* is a metrizable topological space together with a maximal atlas of n-dimensional orbifold charts satisfying certain compatibility conditions. An *orbifold chart* $(U, \tilde{U}, \Gamma_U, \pi_U)$ consists of an open subset $U \subseteq O$, a smooth n-manifold $\tilde{U}$, a finite subgroup $\Gamma_U \subset \mathrm{Diff}(\tilde{U})$, and a continuous map $\pi_U : \tilde{U} \to U$ inducing a homeomorphism $\tilde{U}/\Gamma_U \overset{\cong}{\to} U$.

Any two charts $(U_i, \tilde{U}_i, \Gamma_{U_i}, \pi_{U_i})$, $i = 1, 2$, must be compatible in the following sense: If $\tilde{x}_i \in \tilde{U}_i$ are points with $\pi_{U_1}(\tilde{x}_1) = \pi_{U_2}(\tilde{x}_2)$, then there exists a diffeomorphism $\tilde{\phi} : \tilde{V}_1 \to \tilde{V}_2$ of open neighborhoods $\tilde{V}_i$ of the $\tilde{x}_i$ with $\pi_{U_2} \circ \tilde{\phi} = \pi_{U_1}$. Finally, the charts must *cover M*.

We will denote by $|O|$ the topological space underlying the orbifold O.

More generally, we define smooth n-dimensional orbifolds O *with boundary* by allowing the chart domains $\tilde{U}$ to be smooth n-manifolds with boundary.

The *boundary ∂O*, respectively, the *interior* of O consist of those points whose preimages in the chart domains are boundary, respectively, interior points. Since the local coordinate changes $\tilde{\phi}$ are smooth, they preserve boundaries. Consequently, ∂O is a closed subset and inherits a structure as a smooth $(n-1)$-orbifold without boundary. The boundary has a collar, i.e. ∂O has an open neighborhood in O diffeomorphic to the product orbifold $\partial O \times [0, 1)$ where $[0, 1)$ is to be understood as a 1-manifold with boundary.

Note that the local coordinate change $\tilde{\phi}$ in the above definition must be equivariant with respect to an isomorphism $Stab_{\Gamma_{U_1}}(\tilde{x}_1) \to Stab_{\Gamma_{U_2}}(\tilde{x}_2)$, i.e. the local actions of the stabilizers $Stab_{\Gamma_{U_i}}(\tilde{x}_i)$ near $\tilde{x}_i$ are conjugate. This is due to the fact that smooth orbit equivalences

between effective finite smooth group actions on connected manifolds are conjugacies.

Given a point $x \in O$, we can choose a chart $(U, \tilde{U}, \Gamma_U, \pi_U)$ with $x \in U$ and a preimage $\tilde{x} \in \pi_U^{-1}(x)$. By our previous remark, the germ of the action of $\Gamma_x := Stab_{\Gamma_U}(\tilde{x})$ near $\tilde{x}$ is independent of these choices, and so is its linearisation $\Gamma_x \curvearrowright T_{\tilde{x}}\tilde{U}$. We may hence regard Γ_x as a subgroup of $O(n)$ well-defined up to conjugacy; it is called the *local group* of O at x. We will call the quotient $T_{\tilde{x}}\tilde{U}/\Gamma_x$ the orbifold *tangent space* T_xO of O at x. The orbifold O is called *locally orientable* at the point x if $\Gamma_x \subset SO(n)$.

A point $x \in O$ is said to be *regular* if its local group Γ_x is trivial, and *singular* otherwise. The subset O^{sing} of singular points is the *singular locus*. (An orbifold is a manifold if and only if all its points are regular, i.e. if $O^{sing} = \emptyset$.) We call the conjugacy class of the local group $\Gamma_x \subset O(n)$ the (singular) *type* of the point x. One observes that, in a chart around x, the fixed point set of Γ_x in $\tilde{U}$ is a submanifold and its connected component through $\tilde{x}$ projects to points in O of the same singular type as x. Hence the equivalence classes of points of the same type inherit structures as smooth manifolds. Together they form a natural *stratification* of the orbifold O. We refer to the union of the d-dimensional strata as the singular *d-stratum* $O^{(d)}$. Note that $\overline{O^{(d)}} \setminus O^{(d)} \subset \bigcup_{k<d} O^{(k)}$. If O has boundary, then the singular strata are manifolds with boundary and one has that $\partial O^{(d)} = (\partial O)^{(d-1)}$.

Clearly, the top-dimensional stratum $O^{(n)}$ consists precisely of the regular points. The singular $(n-1)$-stratum $O^{(n-1)}$ consists of the points with local group $\cong \mathbb{Z}_2$ generated by a hyperplane reflection. Its closure $\partial_{refl}O := \overline{O^{(n-1)}}$ is usually referred to as *reflector boundary* or *silvered boundary*, even though it is not contained in the boundary, $\partial_{refl}O \cap \partial O = \partial_{refl}\partial O$. It consists of the points whose local group contains a hyperplane reflection. We call $O^{(n-1)}$ the regular part of the reflector boundary.

Note that the underlying topological space $|O|$ of the orbifold O contains only partial information on the orbifold structure, e.g. on the singular stratification. For instance, $|O|$ can be a topological manifold, although $O^{sing} \neq \emptyset$.

We define an m-dimensional smooth *suborbifold* of a smooth orbifold to be a subset whose preimages in local charts are smooth m-dimensional submanifolds, cf. [BS87, sec. B]. (This is more restrictive than other definitions used in the literature, cf. e.g. [BMP03, 2.1.3]). Analogously, a subset of an orbifold with boundary is called a *proper* suborbifold if its preimages in local charts are proper (smooth) submanifolds.

A proper codimension-one suborbifold $\Sigma^{d-1} \subset O^d$ is called *two-sided* if it has a product neighborhood of the form $\Sigma \times (-1, 1)$. It is called *locally two-sided* at a point x, if it has such a product neighborhood locally near x. If Σ is not (globally) two-sided then it has a tubular neighborhood of the form $(\Sigma' \times (-1, 1))/\mathbb{Z}_2$ where $\mathbb{Z}_2$ reflects on $(-1, 1)$ and acts by a (possibly trivial) involution on Σ'. (To verify this, take e.g. Σ' as the boundary of a tubular neighborhood (thickening) of Σ.)

When speaking of a codim-zero suborbifold $O' \subset O$ we suppose that the components of $\partial O'$ are either components of ∂O or disjoint from ∂O, i.e. two-sided suborbifolds of $int(O)$.

A continuous map $f : O \to O'$ of smooth orbifolds is called *smooth* (in the orbifold sense) if it lifts locally to an equivariant smooth (in the manifold sense) map of charts, i.e. if the following holds: For any point $x \in O$ exist charts $(U, \tilde{U}, \Gamma_U, \pi_U)$ around x and $(U', \tilde{U}', \Gamma_{U'}, \pi_{U'})$

around $x' = f(x)$, and a smooth local lift $\tilde{f}_U : \tilde{U} \to \tilde{U}'$ of f, $f \circ \pi_U = \pi_{U'} \circ \tilde{f}_U$, which is equivariant with respect to a group homomorphism $\rho : \Gamma_U \to \Gamma_{U'}$. (Note that in general the equivariance of lifts is not automatic; there may exist non-equivariant lifts.)

A smooth map is a *submersion* (*immersion*) if it lifts locally to an equivariant submersion (immersion) of charts. It is a *local diffeomorphism* if it can be locally inverted everywhere by a smooth map, equivalently, if it lifts locally to diffeomorphisms of charts which are equivariant with respect to isomorphisms of local groups. A global *diffeomorphism* is a homeomorphism which is also a local diffeomorphism.

A smooth map $p : O' \to O$ between orbifolds is a *covering* if every point $x \in O$ has an open neighborhood U such that for every connected component U' of $p^{-1}(U)$ exists a chart $(U', \tilde{U}', \Gamma_{U'}, \pi_{U'})$ for O' and a possibly larger finite group Γ_U, $\Gamma_{U'} \subseteq \Gamma_U \subset Diff(\tilde{U}')$, such that $(U, \tilde{U}', \Gamma_U, p \circ \pi_{U'})$ is a chart for O. Coverings induce injective homomorphisms of the local groups which are well-defined up to postcomposition with inner automorphisms.

An orbifold is called *(very) good* if it is (finitely) covered by a manifold; an orbifold which is not covered by a manifold is called *bad*.

Let F be an orbifold without boundary. Following [BMP03, sec. 2.4] we define an *orbifold fiber bundle* or *orbifold fibration* with generic fiber F as a submersion $p : O \to B$ of orbifolds, possibly with boundary, with the following property: For every point $x \in B$ exists a chart $\phi : \tilde{U} \to U$ around x, a smooth operation $\Gamma_x \curvearrowright F$ and a submersion $\sigma : \tilde{U} \times F \to O$ inducing a diffeomorphism between $(\tilde{U} \times F)/\Gamma_x$ (where we divide out the diagonal action) and $p^{-1}(U)$ such that $p \circ \sigma = \phi \circ \pi_{\tilde{U}}$. In the case with boundary we require that $p^{-1}(\partial B) = \partial O$; then p restricts over the boundary to the orbifold fiber bundle $p_{|\partial O} : \partial O \to \partial B$. Note that orbifold coverings are (the same as) orbifold fiber bundles with 0-dimensional fiber.

We say that a compact orbifold *fibers* if it is the total space of an orbifold fibration whose base and fiber have strictly positive dimension and whose generic fiber is a closed orbifold.

An orbifold is called *spherical (discal, toric, solid toric)* if it is diffeomorphic to the quotient of a round sphere S^n (a closed unit disc D^n, a flat 2-torus, the compact 3-dimensional solid torus $(D^2 \times S^1)$) by a finite isometric group action.

2.1.2 Riemannian and geometric orbifolds

A *Riemannian orbifold* can be defined as a smooth orbifold together with compatible Riemannian metrics on the local uniformizations $\tilde{U}$, i.e. such that the operations $\Gamma_U \curvearrowright \tilde{U}$ and the local coordinate changes $\tilde{\phi}$ are isometric.

Notions from Riemannian geometry like lengths of curves, path metric, geodesics, exponential map directly generalize to orbifolds via local charts. The natural stratification of O is totally geodesic.

The smooth orbifold structure underlying a Riemannian orbifold is encoded in its metric structure. The singular points in the orbifold sense are also geometric singularities, and a Riemannian orbifold can be defined more directly as a metric space which is locally isometric to the quotient of a Riemannian manifold by a finite group of isometries.

A *geometric structure* on a smooth 3-orbifold is a Riemannian metric which is modelled

on one of the eight 3-dimensional Thurston geometries S^3, $\mathbb{R}^3$, $\mathbb{H}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil, $\widetilde{PSL(2,\mathbb{R})}$ or $Solv$. This means that the Riemannian metrics on the local uniformizations are locally isometric to the respective model space, equivalently, the orbifold Riemannian metric is everywhere locally isometric to a quotient of the model space by a finite group of isometries, cf. sec. 2.1.2.

A compact 3-orbifold is called *geometric* if its interior admits a complete geometric structure. For a closed geometric 3-orbifold the model geometry is unique.

Geometric orbifolds are very good, i.e. finitely covered by manifolds.

2.2 Low-dimensional orbifolds

2.2.1 1-orbifolds

There are two connected closed 1-orbifolds, namely the circle and the *mirrored interval* $\bar{I}$. The latter has as underlying topological space $|\bar{I}|$ the compact interval I. The boundary points of $|\bar{I}|$ are reflector boundary points of $\bar{I}$, $\partial_{refl}\bar{I} = \bar{I}^{(0)} = \partial|\bar{I}|$, but no boundary points; their local groups are $O(1) \cong \mathbb{Z}_2$.

2.2.2 2-orbifolds

We will denote by D^2 the *closed* 2-disc and by $\overline{D}^2$ the closed 2-disc with *reflector boundary*.

Let O^2 be a 2-orbifold, possibly with boundary. A singular point with local group $\cong \mathbb{Z}_p$, $p \geq 2$, acting by rotations is called a *cone point* of order p, respectively in the Riemannian case, with *cone angle* $\frac{2\pi}{p}$. It is an isolated singular point in the interior of O and has a neighborhood diffeomorphic to the disc $D^2(p) := D^2/\mathbb{Z}_p$ with cone point of order p.

A singular point whose local group $\Gamma_x \subset O(2)$ is a dihedral group D_q, $q \geq 2$, is called a *corner vertex* of order q, respectively in the Riemannian case, with angle $\frac{\pi}{q}$.[1] It is an interior point and has a neighborhood diffeomorphic to the sector $V^2(q) := D^2/D_q$. Note that $\partial_{refl}V^2(q) = \partial|V^2(q)|$.

A singular point with local group $\cong D_1 \cong \mathbb{Z}_2$ acting by a reflection (on the disc or half-disc) is a regular reflection boundary point and has a neighborhood diffeomorphic to $V^2(1) := D^2/D_1$. It may be a boundary point, namely one of the points in $\partial O \cap \partial_{refl}O$.

The singular locus $O^{sing} = O^{(1)} \cup O^{(0)}$ consists of the reflector boundary $\partial_{refl}O = \overline{O^{(1)}}$, which contains the set $\overline{O^{(1)}} - O^{(1)} = \overline{O^{(1)}} \cap O^{(0)}$ of corners, and of the set $O^{(0)} - \overline{O^{(1)}}$ of cone points. We call a connected component of $O^{(1)}$ a *reflector edge*. In a corner vertex, locally two reflector edges meet.

Sometimes we will also admit cone points and corner vertices of order 1 which are nothing else than regular interior, respectively, regular reflector boundary points.

A connected component of $\partial|O|$ can be a connected component of ∂O or of $\partial_{refl}O$, or it can

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[1]As a subgroup of $O(2)$, the dihedral group D_q is defined as the isometry group of a regular q-gon. It is generated by the reflections at two lines through the origin with angle $\frac{\pi}{q}$. As an abstract group, it has the presentation $\langle s_1, s_2 | s_1^2 = s_2^2 = (s_1 s_2)^q = 1 \rangle$.

be a chain of consecutive boundary arcs and reflector edges. In the latter case, any two of the boundary arcs are disjoint, but there may be sequences of consecutive reflector edges meeting at corner vertices.

We will use the following notation. For a 2-manifold Σ with boundary, we denote by $\bar{\Sigma}(p_1, \ldots, p_k; q_1, \ldots, q_l)$ a 2-orbifold (without boundary) with underlying space Σ, reflector boundary $\partial\Sigma$, k cone points of orders p_i located in the interior of Σ and l corner vertices of orders q_j lying on $\partial\Sigma$. If there are no cone points ($k = 0$) we write $\Sigma(; q_1, \ldots, q_l)$; if there are no corner vertices ($l = 0$) we will sometimes write briefly $\Sigma(p_1, \ldots, p_k;) =: \Sigma(p_1, \ldots, p_k)$, and also $\Sigma(;) =: \Sigma$. In most instances where we will apply this notation the diffeomorphism type of the 2-orbifold is uniquely determined. Note in particular that $\overline{D}^2(p;) = \overline{D}^2(p)$ is a closed 2-orbifold with a reflector boundary circle.

The *Euler characteristic* of a 2-orbifold $O = \Sigma(p_1, \ldots, p_k; q_1, \ldots, q_l)$ is given by

$$\chi(O) = \chi(\Sigma) - \sum_{i=1}^{k}(1 - \frac{1}{p_i}) - \frac{1}{2}\sum_{j=1}^{l}(1 - \frac{1}{q_j}).$$

We recall the classification of connected closed spherical 2-orbifolds, that is, of quotients of the 2-sphere: S^2, $\mathbb{R}P^2$, $\overline{D}^2$, $S^2(p,p)$, $\mathbb{R}P^2(p)$, $\overline{D}^2(p)$, $\overline{D}^2(;p,p)$, $S^2(2,2,p)$, $S^2(2,3,3)$, $S^2(2,3,4)$, $S^2(2,3,5)$, $\overline{D}^2(;2,2,p)$, $\overline{D}^2(;2,3,3)$, $\overline{D}^2(;2,3,4)$, $\overline{D}^2(;2,3,5)$, $\overline{D}^2(2;p)$ and $\overline{D}^2(3;2)$ with $p \geq 2$.

The list of connected closed flat 2-orbifolds, i.e. of quotients of the 2-torus is: T^2, K^2, Ann^2, Möb^2, $S^2(2,3,6)$, $S^2(2,4,4)$, $S^2(3,3,3)$, $S^2(2,2,2,2)$, $\mathbb{R}P^2(2,2)$, $\overline{D}^2(;2,3,6)$, $\overline{D}^2(;2,4,4)$, $\overline{D}^2(;3,3,3)$, $\overline{D}^2(;2,2,2,2)$, $\overline{D}^2(4;2)$, $\overline{D}^2(3;3)$, $\overline{D}^2(2;2,2)$ and $\overline{D}^2(2,2)$.

All connected bad 2-orbifolds are closed and can be obtained by gluing two non-diffeomorphic discal 2-orbifolds along their boundaries. They are diffeomorphic to $S^2(p)$, $S^2(p,q)$, $\overline{D}^2(;p)$ or $\overline{D}^2(;p,q)$ with $2 \leq p < q$.

Proposition 2.1. *A connected closed 2-orbifold admits a Riemannian metric with nonnegative sectional curvature if and only if it has Euler characteristic ≥ 0 if and only if it is spherical, flat or bad.*

2.2.3   3-orbifolds

Let O^3 be a 3-orbifold, possibly with boundary. We first discuss the local structure of the singular locus.

If $x \in O^{(1)}$, then the local group $\Gamma_x \subset O(3)$ fixes a line, i.e. $\Gamma_x \cong \mathbb{Z}_p$ or D_p with $p \geq 2$ and $S^2/\Gamma_x \cong S^2(p,p)$ or $\overline{D}^2(;p,p)$. We call the connected component of $O^{(1)}$ containing x a *singular edge*, respectively, *reflector edge* (or circle) of order p. In the reflector case $\Gamma_x \cong D_p$, locally two reflector faces, that is, components of $O^{(2)}$ meet at the edge. The boundary points of the singular and reflector edges (i.e. of their underlying 1-manifolds) are the cone points and corner vertices of the 2-orbifold ∂O.

If $x \in O^{(0)}$, then we call x a *singular vertex*. In this case, $\Gamma_x \subset O(3)$ has no non-trivial fixed vector and S^2/Γ_x is a spherical 2-orbifold with diameter $< \pi$, i.e. isometric to $\mathbb{R}P^2$, $\mathbb{R}P^2(p)$, $\overline{D}^2(p)$, $S^2(2,2,p)$, $S^2(2,3,3)$, $S^2(2,3,4)$, $S^2(2,3,5)$, $\overline{D}^2(;2,2,p)$, $\overline{D}^2(;2,3,3)$, $\overline{D}^2(;2,3,4)$, $\overline{D}^2(;2,3,5)$, $\overline{D}^2(2;p)$ or $\overline{D}^2(3;2)$ with $p \geq 2$. We have $x \in \partial_{refl}O$ if and only if

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$\partial_{refl}(S^2/\Gamma_x) \neq \emptyset$. The cone points and corner vertices of S^2/Γ_x correspond to singular edges emanating from x.

2.2.4   3-orbifolds with nonnegative sectional curvature

Furthermore, we recall the classification of noncompact 3-orbifolds (without boundary) admitting complete Riemannian metrics of nonnegative sectional curvature. It follows from an orbifold version of the Soul Theorem which states that a complete noncompact Riemannian orbifold with sectional curvature ≥ 0 contains a totally convex and totally geodesic closed suborbifold, a so-called *soul*, and the orbifold is diffeomorphic to the normal bundle of the soul.

Proposition 2.2. *Every connected complete noncompact Riemannian 3-orbifold with sectional curvature ≥ 0 is diffeomorphic to one of the following:*

1. *Quotients $\mathbb{R}^3/\Gamma$ for finite subgroups $\Gamma \subset O(3)$. (The soul is a point.)*

2. *Bundles over S^1 with fiber $\mathbb{R}^2/\Gamma'$ for finite subgroups $\Gamma' \subset O(2)$. (The soul is a circle.)*

3. *A 3-orbifold arising from $D^2(p) \times [-1, 1]$, $p \geq 1$, by gluing each of the boundary components $D^2(p) \times \{\pm 1\}$ either to itself via a half-rotation or reflection or by making it a reflector boundary component. There are six such orbifolds. (The soul is a mirrored interval.)*

4. *A 3-orbifold arising from $V^2(p) \times [-1, 1]$, $p \geq 1$, by gluing each of the boundary components $V^2(p) \times \{\pm 1\}$ either to itself via the reflection at its bisector or by making it a reflector boundary component. There are three such orbifolds. (The soul is a mirrored interval.)*

5. *Products $\Sigma^2 \times \mathbb{R}$ for closed 2-orbifolds Σ^2 with Euler characteristic ≥ 0. (The soul is 2-dimensional.)*

6. *A 3-orbifold arising from $\Sigma^2 \times [0, 1)$, where Σ^2 is a closed 2-orbifold with Euler characteristic ≥ 0, by gluing $\Sigma^2 \times \{0\}$ to itself by a non-trivial involution. (The soul is 2-dimensional.)*

We observe for future reference that all orbifolds occuring in the proposition are 3-discal if and only if the soul is a point, and solid toric if and only if the soul is 1-dimensional.

Moreover, we note an alternative constructions for noncompact complete 3-orbifolds with sec ≥ 0 and soul a mirrored interval: Such orbifolds can also be obtained by starting with two quotients of the 3-ball, each with boundary $\mathbb{R}P^2(p)$, $S^2(2, 2, p)$ or $\bar{D}^2(p)$ for some fixed $p \geq 1$, and glueing them together along a closed pointed disc $D^2(p)$ contained in both boundaries. The (interior of the) resulting 3-orbifolds are then diffeomorphic to those arising from $D^2(p) \times [-1, 1]$ as in *3.)*.

Similarly, we could start with two quotients of the 3-ball, each with boundary $\bar{D}^2(; 2, 2, p)$ or $\bar{D}^2(2; p)$ for some fixed $p \geq 1$, and then identify two sectors $V^2(p)$. In this way, we obtain precisely the orbifolds arising from $V^2(p) \times [-1, 1]$ as in *4.)*.

2.3 Fibrations and decompositions

2.3.1 Fibered 3-orbifolds

An *orbifold Seifert fibration* is an orbifold fibration $p : O^3 \to B^2$ with 3-dimensional total space O, 2-dimensional base B and 1-dimensional closed connected generic fiber F, i.e. F is the circle S^1 or the mirrored interval $\overline{I}$. A *Seifert orbifold* is a 3-orbifold admitting a Seifert fibration.

Every fiber has a neighborhood which is fiber preserving diffeomorphic to a solid toric orbifold equipped with a canonical Seifert fibration. More precisely, suppose that $x \in B$ is a point in the base and let Γ_x be its local group. Then the fiber $p^{-1}(x)$ has a saturated neighborhood of the form $(D^2 \times F)/\Gamma_x$ with the natural fibration $(D^2 \times F)/\Gamma_x \to D^2/\Gamma_x$. The action $\Gamma_x \curvearrowright D^2$ is effective, whereas the action $\Gamma_x \curvearrowright F$ is in general not. Fibers in the boundary have similar model neighborhoods. A classification of Seifert orbifolds, locally and globally, has been given in [BS85].

Seifert fibrations of solid toric 3-orbifolds as well as 1-dimensional fibrations of their toric boudaries are in general not unique. The next result describes which fibrations of the boundary extend to Seifert fibrations. Let $V \cong (D^2 \times S^1)/\Gamma$ be a solid toric 3-orbifold. We call a 1-dimensional fibration of ∂V *horizontal* if it is isotopic to the fibration $\partial V \to S^1/\Gamma$.

Lemma 2.3. *A 1-dimensional fibration of the boundary ∂V of a solid toric orbifold V extends to a Seifert fibration of V if and only if it is not horizontal.*

Proof. This is a consequence of the fact that 1-dimensional fibrations of closed flat 2-orbifolds can be isotoped to be geodesic. $\qquad\square$

All compact Seifert 3-orbifolds without bad 2-suborbifolds are geometric, see [Th97, ch. 3] and [BMP03, 2.4]. More precisely, a connected closed Seifert orbifold without bad 2-suborbifolds admits a geometric structure modelled on a unique Thurston geometry different from hyperbolic and solvgeometry (i.e. on $S^2 \times \mathbb{R}^1$, S^3, $\mathbb{R}^3$, Nil, $\mathbb{H}^2 \times \mathbb{R}^1$, or $\widetilde{PSL(2,\mathbb{R})}$). A solid toric 3-orbifold admits, depending on its topological type, geometric structures modelled on some or all of the six contractible model geometries.

A non-solid toric connected compact Seifert orbifold with nonempty boundary contains no bad 2-suborbifolds and admits an $\mathbb{H}^2 \times \mathbb{R}$- or $\mathbb{R}^3$-structure. (If it admits an $\mathbb{R}^3$-structure then also an $\mathbb{H}^2 \times \mathbb{R}$-structure.) In fact, it can be geometrized in a stronger sense; namely, it admits a Riemannian metric with totally geodesic boundary locally modelled on either $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{R}^3$.

A *toric fibration* of a 3-orbifold is an orbifold fibration whose generic fiber is a toric 2-orbifold. (Fibrations with 2-dimensional fibers of other topological types will play no role in this text.)

Connected 3-orbifolds admitting toric fibrations are geometric with one of the three model geometries $\widetilde{PSL(2,\mathbb{R})}$, Nil or $\mathbb{R}^3$. Those with nonempty boundary admit euclidean metrics with totally geodesic boundary and complete $\mathbb{R}^3$-structures on their interior. They are diffeomorphic to $T \times [-1,1]$ or $(T \times [-1,1])/\mathbb{Z}_2$ with a toric 2-orbifold T and, in the latter case, with $\mathbb{Z}_2$ acting by a reflection on $[-1,1]$. Unlike in the manifold case, they are not always Seifert. This is due to the fact that, whereas a 2-torus or Klein bottle admits (infinitely many, respectively, two)

circle fibrations, not all toric 2-orbifolds admit 1-dimensional orbifold fibrations. (Compare the discussion in [Du88].)

2.3.2 2-suborbifolds in 3-orbifolds

A 3-orbifold O is called *irreducible* if it does not contain any bad 2-suborbifold and if every two-sided spherical 2-suborbifold bounds a discal 3-suborbifold.

It is called *(topologically) atoroidal* if every incompressible two-sided toric 2-suborbifold $\Sigma \subset O$ is *boundary parallel*, i.e. bounds a collar neighborhood $\cong \Sigma \times [0,1]$ of a boundary component $\cong \Sigma$.

Let O be a 3-orbifold and let $\Sigma \subset O$ be a proper 2-suborbifold. A *compressing discal 2-suborbifold* or *compression disc* for Σ is a discal 2-suborbifold $D \subset O$ which intersects Σ transversally in $\partial D = D \cap \Sigma$ such that ∂D does not bound a discal 2-suborbifold in Σ. (If D is one-sided, we understand this to mean that splitting the connected component of Σ containing ∂D along ∂D does not yield a discal 2-orbifold. Anyway, one-sided compression discs can be replaced by two-sided ones by passing to the boundary of a tubular neighborhood.) Note that a spherical 2-suborbifold has no compression discs because every closed 1-suborbifold of a spherical 2-orbifold bounds a discal 2-suborbifold.

A *compression* of Σ is either a discal 3-suborbifold whose boundary is a component of Σ or a compression disc for Σ. If Σ admits a compression then it is called *compressible*, and otherwise *incompressible*.

Thus a 3-orbifold is irreducible if it contains no bad 2-suborbifolds and if all two-sided spherical 2-suborbifolds are compressible.

The notion of incompressibility is particularly useful in the irreducible case because then the position e.g. of closed 2-suborbifolds Σ relative to incompressible 2-suborbifolds Σ_{inc} can be simplified by isotopies. Namely, it can be acheived that Σ_{inc} divides Σ into non-discal components.

Discal 3-orbifolds are irreducible. This is formulated but not proved in [BMP03, Thm. 3.1]. A proof can be found in [DL09, 2.4].

More generally, every closed 2-suborbifold of a discal 3-orbifold is compressible. For non-spherical suborbifolds this follows from the Equivariant Loop Theorem [MY80], cf. [BMP03, Thm. 3.6]. We will use it only for toric 2-suborbifolds.

2.3.3 Decompositions of 3-orbifolds along 2-suborbifolds

We suppose in the following that O is a compact 3-orbifold.

Let $\mathcal{F}$ be a finite family of disjoint two-sided closed 2-suborbifolds $\Sigma_j \subset int(O)$. The operation of removing from O an open tubular neighborhood of $\cup_j \Sigma_j$ is called *splitting* O along the Σ_j. We call the splitting spherical (toric, incompressible) if all Σ_j are spherical (toric, incompressible). We will refer to the Σ_j as *splitting 2-suborbifolds*.

A *connected sum decomposition* of O or a *surgery* on O is performed by first splitting O along a family of spherical 2-suborbifolds and then filling discal 3-orbifolds into the additional

spherical boundary components created by the splitting. Conversely, O is called a *connected sum* of the 3-orbifolds resulting from this decomposition. Note that we allow connected sums of connected orbifolds (components) with themselves.

The following result reduces the study of compact 3-orbifolds without bad 2-suborbifolds to the study of irreducible ones. It is due to Kneser [Kn29] in the manifold case, see [BMP03, 3.3] for a proof in the case of orientable orbifolds. The argument given there also extends to the nonorientable case.

Theorem 2.4 (Spherical decomposition). *A compact 3-orbifold without bad 2-suborbifolds can be decomposed by surgery into finitely many irreducible compact 3-orbifolds.*

Let us now consider toric splittings.

Lemma 2.5. *Suppose that O is split along a toric family $\mathcal{T}$ into compact pieces O_i. Then O is irreducible and $\mathcal{T}$ is incompressible (in O) if and only if all pieces O_i are irreducible and for each piece O_i the portion $\partial O_i - \partial O$ of its boundary corresponding to $\mathcal{T}$ is incompressible (in O_i). Moreover, if in this situation all boundary components of the O_i are incompressible, then O has incompressible boundary.*

Proof. The standard proof in the manifold case carries over. The "only if" direction uses the fact that toric 2-suborbifolds of discal 3-orbifolds are always compressible. $\square$

There is a canonical splitting of irreducible compact 3-orbifolds along incompressible toric suborbifolds. It is due to Jaco, Shalen and Johannson in the manifold case and has be extended to orbifolds by Bonahon and Siebenmann [BS87], see also [BMP03, 3.3 and 3.15].

Theorem 2.6 (JSJ-splitting). *An irreducible compact 3-orbifold admits an incompressible toric splitting into components each of which is atoroidal or Seifert fibered (or both). A minimal such splitting is unique up to isotopy.*

We will also consider a class of toric splittings with weaker properties. Following Waldhausen's definition [Wa67] in the manifold case, we define a *graph splitting* of a compact 3-orbifold with toric boundary to be a (not necessarily incompressible) toric splitting into pieces which admit orbifold fibrations with 1- or 2-dimensional closed fibers. Moreover, the 2-dimensional fibers are required to be toric. We will refer to the pieces with 2-dimensional fibrations as pieces *with toric fibrations*. A 3-orbifold admitting a graph splitting is called a *graph orbifold*. Briefly, it is a 3-orbifold which can be "cut up into fibered pieces".

Connected compact 3-orbifolds with toric fibrations and nonempty boundaries are diffeomorphic to $T \times [-1, 1]$ or $(T \times [-1, 1])/\mathbb{Z}_2$ with a toric 2-orbifold T and, in the latter case, with $\mathbb{Z}_2$ acting by a reflection on $[-1, 1]$. Unlike in the manifold case, they are not always Seifert. This is due to the fact that, whereas a 2-torus or Klein bottle admits (infinitely many, respectively, two) circle fibrations, not all toric 2-orbifolds admit 1-dimensional orbifold fibrations, as already discussed above. Hence, in the orbifold case a graph splitting may comprise non-Seifert pieces.

Seifert orbifolds with discal base orbifold are solid toric. All other connected Seifert orbifolds with nonempty boundary have base orbifolds of Euler characteristic $\chi \leq 0$ and admit

nonpositively curved Riemannian metrics with totally geodesic boundary. (These metrics can be modelled on $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{R}^3$, depending on whether $\chi < 0$ or $\chi = 0$.) The 3-orbifolds with toric fibrations and nonempty boundary admit flat metrics with totally geodesic boundary. The existence of these geometric structures on the non-solid toric pieces of a graph splitting implies that they are irreducible and have incompressible boundaries.[2] Moreover, the pieces with toric fibrations and nonempty boundaries are atoroidal.

If in a non-trivial graph splitting of O occur no solid toric pieces, then all pieces are irreducible atoroidal or Seifert orbifolds with nonempty incompressible toric boundaries. Hence O is irreducible with incompressible toric boundary and the splitting is incompressible, compare Lemma 2.5. In particular, a minimal incompressible graph splitting of an irreducible compact connected 3-orbifold with incompressible toric boundary is *canonical* up to isotopy because it coincides with the JSJ-splitting, unless the orbifold admits a toric fibration over a closed 1-orbifold (in which case it is geometric). Indeed, suppose that a nontrivial minimal incompressible graph splitting were not minimal as a splitting into atoroidal and Seifert components. Then for some splitting toric 2-suborbifold T the union of the (one or two) components adjacent to it cannot be Seifert and must therefore be atoroidal. The definition of atoroidality then implies that one of these components is $\cong T \times [0, 1]$, contradicting the minimality of the graph splitting.

A *geometric splitting* of an irreducible compact connected 3-orbifold O is an incompressible toric splitting into geometric pieces, i.e. into irreducible compact 3-orbifolds whose interiors admit complete geometric structures. We refer to the components of the splitting as *geometric pieces*. Note that if O itself is not geometric, then the pieces have nonempty boundaries and admit geometric structures modelled on $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{R}^3$. In particular, a nontrivial incompressible graph splitting of O is a geometric splitting into pieces admitting $\mathbb{H}^2 \times \mathbb{R}$- or $\mathbb{R}^3$-structures.

A compact 3-orbifold is said to be *decomposable into geometric pieces* or to *satisfy Thurston's Geometrization Conjecture* if it can be decomposed by surgery into irreducible compact connected 3-orbifolds which are geometric or admit a geometric splitting, cf. [BMP03, 3.7].

2.4   From graph splittings to geometric decompositions

Graph splittings of compact 3-orbifolds have fairly weak properties and are in particular far from being unique. In this section we show that a graph splitting can be improved to a geometric decomposition. The manifold case of this discussion is due to Waldhausen [Wa67].

Theorem 2.7. *Suppose that O is a compact connected 3-orbifold with toric boundary which admits a graph splitting.*

If O is irreducible, then it is either solid toric or has incompressible toric boundary. In the latter case, it is geometric with model geometry different from $S^2 \times \mathbb{R}$ and H^3, or it admits an incompressible graph splitting (and hence a JSJ-splitting without hyperbolic components).

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[2]For the irreducibility one uses the fact that discal 3-orbifolds are irreducible. Namely, consider an embedded 2-sphere $S \subset \tilde{P}$ in the universal cover of such a piece P preserved by a finite group Γ_S of isometries. With the Hadamard-Cartan Theorem it follows that S is contained in a Γ_S-invariant closed ball on which the Γ_S-action is standard. Hence the action on the ball bounded by S is also standard.

If O is not irreducible, it can be decomposed by surgery into irreducible orbifolds of this kind.

Proof. Consider a graph splitting of O along a toric family $\mathcal{T}$.

If $\mathcal{T} = \emptyset$, then O is solid toric, or it is closed and admits an $S^2 \times \mathbb{R}$-structure, or it is irreducible with (possibly empty) incompressible boundary and admits a geometric structure modelled on one of the six geometries different from $S^2 \times \mathbb{R}$ and H^3. If O admits an $S^2 \times \mathbb{R}$-structure, it can be decomposed by surgery along a spherical cross section into one or two spherical 3-orbifolds.

If $\mathcal{T} \neq \emptyset$, then the pieces of the splitting have nonempty boundary. If one of the pieces P is of the form $T \times [0, 1]$ with a toric 2-orbifold T, we may reduce $\mathcal{T}$ by erasing one of the components of ∂P, unless P is the only piece. In the latter case, O fibers over the circle and is geometric (with model geometry $\mathbb{R}^3$, *Nil* or *Solv*).

If no solid toric piece occurs in the graph splitting (and if $\mathcal{T} \neq \emptyset$), then O is irreducible with incompressible boundary and the graph splitting is incompressible, cf. Lemma 2.5.

If there is a solid toric piece and if adjacent to it there is another solid toric piece or a one-ended piece with toric fibration, i.e. a piece diffeomorphic to $(T \times [-1, 1])/\mathbb{Z}_2$ with T toric and $\mathbb{Z}_2$ reflecting on $[-1, 1]$, then O is closed and geometric with model geometry S^3 or $S^2 \times \mathbb{R}$.

Since in all other cases we are done or can reduce the splitting by removing a component from $\mathcal{T}$, we assume that at least one solid toric piece $V_0 \cong (D^2 \times S^1)/\Gamma$ occurs in the graph splitting and that adjacent to V_0 there is a non-solid toric Seifert piece S. We may further assume that all pieces with toric fibrations are one-ended, i.e. diffeomorphic to $(T \times [-1, 1])/\mathbb{Z}_2$ with T toric and $\mathbb{Z}_2$ reflecting on $[-1, 1]$.

We denote by $q : V_0 \cong (D^2 \times S^1)/\Gamma \to S^1/\Gamma$ the fibration of V_0 by discal cross sections. Let $K \triangleleft \Gamma$ be the kernel of the action $\Gamma \curvearrowright S^1$. Via the action $K \curvearrowright D^2$ we may regard K as a subgroup $K \subset O(2)$. The generic discal cross section of V_0 is $\cong D^2/K$. Let $p : S \to B$ denote the Seifert fibration of S. The base B is a non-discal 2-orbifold with nonempty boundary. Let $T_0 = \partial V_0 = S \cap V_0 \in \mathcal{T}$ denote the toric 2-suborbifold separating V_0 and S, and $\partial_0 B = p(T_0)$ the boundary component of B corresponding to T_0. It is either a circle or an arc connecting two points of $\partial_{refl} B \cap \partial B$.

If the fibrations $p|_{T_0}$ and $q|_{T_0}$ of T_0 by closed 1-orbifolds are not isotopic, then the Seifert fibration p can be extended over V_0, compare Lemma 2.3, i.e. $S \cup V_0$ is Seifert and we reduce the graph splitting by removing the component T_0 from $\mathcal{T}$. (The Seifert piece "swallows" the adjacent solid toric piece.)

Otherwise, if $p|_{T_0}$ and $q|_{T_0}$ are isotopic, we may assume that they agree, i.e. that the discal cross sections of V_0 fill in Seifert fibers. We then have the identification $\partial_0 B \cong S^1/\Gamma$. In this situation we find the following class of two-sided spherical 2-suborbifolds adapted to the graph structure. Let $\alpha \subset B - B^{sing}$ be a properly embedded arc with endpoints in $\partial_0 B - \partial_{refl} B$. It yields the spherical 2-suborbifold $\Sigma_\alpha \subset S \cup V_0$ obtained by taking $p^{-1}(\alpha) \subset S$ and attaching to it the pair of discal 2-suborbifolds $q^{-1}(\partial \alpha) \subset V_0$. Hence $\Sigma_\alpha \cong S^2/K$ where we extend the action of $K \subset O(2)$ to $\mathbb{R}^3$ using the canonical embedding $O(2) \subset O(3)$. If $\partial_{refl} B \neq \emptyset$ we get another similar class of spherical 2-suborbifolds by taking embedded arcs $\hat{\alpha}$ connecting a regular boundary point on $\partial_0 B$ to an interior point of a reflector edge. In this case we have

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$\Sigma_{\hat{\alpha}} \cong S^2/\hat{K}$ where $\hat{K} \subset O(2)$ is an index two extension of K such that the elements in $\hat{K} - K$ switch the poles $(0, 0, \pm 1)$ of S^2.

We note that the discal 3-orbifold D^3/K has a decomposition into a Seifert and a solid toric piece analogous to the decomposition of $S \cup V_0$. Indeed, take an $O(2)$-invariant decomposition of D^3 into a tubular neighborhood of the equator $S^1 \times \{0\}$ and the complement of this neighborhood. By dividing out K, one sees that D^3/K is obtained by attaching the cylinder $(D^2/K) \times [-1, 1]$ to a product fibration with fiber S^1/K and base a bigon (topological disc) such that the fibrations of the boundaries match. Thus we may split O along Σ_α, fill in copies of D^3/K into the two spherical boundary components resulting from the splitting to obtain a (possibly disconnected) 3-orbifold O_α, and extend the graph splitting to all of O_α by attaching copies of $(\partial D^2/K) \times [-1, 1]$ to the pieces of T_0. The effect on the base of the Seifert piece is that B is split along α and two bigons are attached along the copies of α. The discal 3-orbifold $D^3/\hat{K}$ is obtained analogously by attaching the cylinder quotient $(D^2 \times [-1, 1])/\hat{K}$, where $\hat{K}/K$ acts on $[-1, 1]$ by a reflection, to a Seifert manifold with base orbifold a triangular disc Δ whose boundary consists of two boundary arcs and one reflector boundary arc. When performing the connected sum decomposition (surgery) of O along $\Sigma_{\hat{\alpha}}$ and extending the graph splitting over $O_{\hat{\alpha}}$, the effect on the base of the Seifert piece is that B is again split along $\hat{\alpha}$, but this time copies of Δ are attached along the copies of $\hat{\alpha}$.

If $\mathcal{A}$ is a finite system of disjoint properly embedded arcs α_i and $\hat{\alpha}_j$ in B as above, we denote the result of performing simultaneous surgeries along the system of spherical 2-suborbifolds Σ_{α_i} and $\Sigma_{\hat{\alpha}_j}$ by $O_\mathcal{A}$ and equip it with an induced graph splitting along a toric family $\mathcal{T}_\mathcal{A}$ as explained. The components of $\mathcal{T}$ different from T_0 correspond to components of $\mathcal{T}_\mathcal{A}$, whereas T_0 may split up into several components. Furthermore, we denote by $S_\mathcal{A} \subset O_\mathcal{A}$ the Seifert suborbifold corresponding to S and by $B_\mathcal{A}$ the base of its Seifert fibration.

Now we choose the system of arcs $\mathcal{A}$ so that they split B into pieces as simple as possible. After making a suitable choice, every connected component B' of $B_\mathcal{A}$ is diffeomorphic to one of the compact 2-orbifolds in the following list:

- *Ann* or the quadrangle Q bounded by two boundary and two reflector edges occuring in alternating order (a quotient of *Ann* by an involution);

- $\overline{D}^2(p)$ or $\overline{V}^2(p)$ with $p \geq 1$.

Lemma 2.8. *Suppose that the compact 3-orbifold O' has a decomposition $O' = S' \cup V'$ along a toric 2-suborbifold $T' = S' \cap V'$ into a Seifert piece S' and a solid toric piece V' such that the discal cross sections of V' fill in Seifert fibers of S'. If B' belongs to the above list, then O' is either spherical or solid toric.*

Proof. If B' splits as the product of the compact interval and a connected closed 1-orbifold, i.e. if B' is the annulus or the quadrangle, then $O' = S' \cup V' \cong V'$ is solid toric.

If B' is discal, then S' is also solid toric and has the form $S' \cong (D^2 \times F')/\Gamma'$ with faithful action $\Gamma' \curvearrowright D^2$ and generic Seifert fiber F'. Let Δ' denote the discal cross section of V'. Using an identification $\partial \Delta' \cong F'$, we form the closed 3-orbifold $\hat{O}'$ by gluing $D^2 \times F'$ and $\partial D^2 \times \Delta'$ canonically along their boundaries. We extend the Γ'-action from $D^2 \times F'$ to $\hat{O}'$ by choosing

an extension of the Γ'-action on F' to Δ'. There exists a Γ'-invariant spherical structure on $\hat{O}'$. Hence $O' \cong \hat{O}'/\Gamma'$ is spherical. $\square$

Proof of Theorem 2.7 continued. As a consequence of Lemma 2.8, the splitting of $O_\mathcal{A}$ along the subfamily of $\mathcal{T}_\mathcal{A}$ consisting of those toric 2-suborbifolds, which correspond to the components of $\mathcal{T}$ different from T_0, is still a graph splitting. (The pieces resulting from splitting V_0 swallow the corresponding adjacent pieces of S.)

Our discussion yields so far: O is irreducible and satisfies the conclusion of the theorem, or O is closed and admits an $S^2 \times \mathbb{R}$-structure, or O can be decomposed by surgery into components which admit graph splittings along strictly fewer toric 2-suborbifolds.

By repeating this process finitely many times, it follows that O can be decomposed by surgery into irreducible orbifolds satisfying the conclusion of the theorem. In particular, the assertion holds if O is not irreducible. If O is irreducible, then O is diffeomorphic to one of the components arising from the surgery, and the assertion holds as well. $\square$

Corollary 2.9. *A compact connected 3-orbifold with toric boundary which admits a graph splitting is decomposable into geometric pieces (and hence satisfies Thurston's Geometrization Conjecture).*

Remark 2.10. Let O be as in Theorem 2.7. Then the boundary of O is incompressible if and only if no solid toric components occur in the surgery decomposition of O. Indeed, suppose that V is a solid toric component. Then there exists a finite family of disjoint embedded discal 3-suborbifolds $B_i \subset V$ such that $V - \cup_i B_i$ embeds into O. There exists a compression disc for ∂V in V avoiding the B_i, and hence a compression disc for $\partial V \subset \partial O$ in O. Conversely, suppose that ∂O is compressible and consider a compression disc Δ. Using the property that O contains no bad 2-suborbifolds, we can make Δ step by step disjoint from the family of spherical 2-suborbifolds along which the surgery is performed until Δ is contained in an irreducible component. This component must be solid toric.

3 Coarse stratification of roughly ≤ 2-dimensional Alexandrov spaces

3.1 Preliminaries

For background on spaces with curvature bounded below we refer to the basic text [BGP92] and to the material in [BBI01, ch. 10].

By a *segment* we mean more precisely a distance minimizing geodesic segment. Given two points x and y, then xy denotes one of the possibly several segments connecting these points.

3.1.1 Alexandrov balls

All arguments from Alexandrov geometry used in this paper will be local and accordingly work in an appropriate class of local Alexandrov spaces.

Definition 3.1 (Alexandrov ball). An *Alexandrov ball* of curvature $\geq \kappa$ is a local Alexandrov space with curvature $\geq \kappa$ of the form $X = B(x, \rho)$, $\rho > 0$, with the additional properties that the closed balls $\overline{B}(x, r)$ for $r \in (0, \rho)$ are metrically complete, and that for any two points $y, z \in X$ with $d(y, z) + d(x, y) + d(x, z) < 2\rho$ exists a segment yz joining y and z.

The first property can be viewed as metrical completeness "up to radius $< \rho$" and the second is a global form of the length space condition. We call x the *center* of the Alexandrov ball $B(x, \rho)$ and the minimal number $r \in [0, \rho]$ such that $\overline{B}(x, r) = B(x, \rho)$ its *radius* (with respect to x). An Alexandrov space may be regarded as an Alexandrov ball with infinite radius.

For any pair of points in $B(x, \frac{\rho}{2})$ or, more generally, in a ball $B(y, r)$ with $d(x, y) + 2r \leq \rho$ exists a segment connecting them. For any triple of vertices in $B(x, \frac{\rho}{3})$ or, more generally, in a ball $B(y, r)$ with $d(x, y) + 3r \leq \rho$ geodesic triangles exist and they satisfy triangle comparison due to the version of Toponogov's Theorem for Alexandrov spaces, cf. [BGP92, §3].

3.1.2 Strainers and cross sections

Let $X = B(x, R)$ be an Alexandrov ball with curvature ≥ -1. All points occuring in our discussion below are supposed to lie in $B(x, \frac{R}{3})$.

For a (small) constant $\theta > 0$, a θ-straight n-*strainer* of length l $(> l)$ in a point $x \in X$ consists of n pairs of points a_i, b_i at distance l $(> l)$ from x such that $\tilde{\angle}_p(a_i, b_i) \geq \pi - \theta$, $\tilde{\angle}_p(a_i, a_j) \geq \frac{\pi}{2} - \theta$ and $\tilde{\angle}_p(b_i, b_j) \geq \frac{\pi}{2} - \theta$ for all $i \neq j$, and $\tilde{\angle}_p(a_i, b_j) \geq \frac{\pi}{2} - \theta$ for all i, j. (All comparison angles are taken in the hyperbolic plane.) We call the strainer $< \theta$-straight if it is θ'-straight for some $\theta' < \theta$. Compare the definition of *burst points* in [BGP92, §5.2] and the definition of strainers [BBI01, §10.8.2]. We say that a strainer is *equilateral* if all points a_i, b_i have the same distance from x.

Similarly, we define a θ-straight $n\frac{1}{2}$-*strainer* of length l $(> l)$ in x as such an n-strainer together with an additional point a_{n+1} at distance l $(> l)$ from x such that $\tilde{\angle}_p(a_{n+1}, a_i) \geq \frac{\pi}{2} - \theta$ and $\tilde{\angle}_p(a_{n+1}, b_i) \geq \frac{\pi}{2} - \theta$ for all $i \leq n$. We define an *infinitesimal* strainer as a configuration of directions in $\Sigma_x X$ satisfying analogous inequalities.

Due to the monotonicity of comparison angles, the existence of a strainer of length l at x implies the existence of strainers at x of the same type and straightness with any length $l' < l$.

For an n-strainer $(a_1, b_1, \ldots, a_n, b_n)$ in x we put

$$f_i := f_{a_i, b_i} := \frac{1}{2}(d(a_i, \cdot) - d(b_i, \cdot)) - \frac{1}{2}(d(a_i, x) - d(b_i, x))$$

and $f := f_{a_1, b_1, \ldots, a_n, b_n} := (f_1, \ldots, f_n)$, normalized to vanish in x. The functions f_i are 1-Lipschitz. We call the level sets $f^{-1}(t)$ the *cross sections* of the strainer and denote by $\Sigma_{y; a_i, b_i} = f_i^{-1}(f_i(y))$ and $\Sigma_{y; a_1, b_1, \ldots, a_n, b_n} = f^{-1}(f(y))$ the cross sections through the point y.

The (Hausdorff) *dimension* can be characterized in terms of strainers as follows. There exists a constant $\bar{\theta}_{d\frac{1}{2}} > 0$ such that X has dimension $> d$ if and only if some point in X admits a $\bar{\theta}_{d\frac{1}{2}}$-straight $d\frac{1}{2}$-strainer of some positive length. (This in turn is implied by the existence of a $< \bar{\theta}_{d\frac{1}{2}}$-straight infinitesimal $d\frac{1}{2}$-strainer in some point.) These constants need not be extremely small, e.g. $\bar{\theta}_{1\frac{1}{2}}$ may be chosen arbitrarily in $(0, \frac{\pi}{2})$ and $\bar{\theta}_{2\frac{1}{2}}$ in $(0, \frac{\pi}{10})$.

Thus, if a sequence of d-dimensional pointed Alexandrov balls (X_i, x_i) with curvature ≥ -1 *collapses* to a pointed Alexandrov ball (X_∞, x_∞) of dimension $\leq k < d$, $(X_i, x_i) \to (X_\infty, x_\infty)$, then for any radius $r > 0$ the supremum of the lengths of $\bar{\theta}_{k\frac{1}{2}}$-straight $k\frac{1}{2}$-strainers in points of $\overline{B}(x_i, r)$ tends to zero as $i \to \infty$.

3.1.3 Comparing comparison angles

We will need to compare the comparison angles of (equilateral) 1-strainers with respect to different model spaces of constant curvature with curvature values in the interval $[-1, 0]$. We observe that for a triangle with fixed side lengths comparison angles increase monotonically with the comparison curvature value. We are interested in 1-strainers of bounded length.

Consider a triangle Δ in euclidean space with two sides of length 2 and angle $\pi - \theta$ between them. We define the angle $\alpha(\theta)$ by letting $\pi - \alpha(\theta) < \pi - \theta$ be the comparison angle of Δ with respect to hyperbolic space of curvature -1, i.e. the corresponding angle of a triangle in hyperbolic space with the same side lengths as Δ.

Lemma 3.2. *The function $\alpha(\theta)$ is differentiable in $\theta = 0$ with $\alpha'(0^+) = (2 \coth 2)^{\frac{1}{2}} < \frac{3}{2}$.*

Proof. Let $l = 4 - h$ denote the length of the third side of Δ, i.e. $l = 4 \cos \frac{\theta}{2}$ and $h = \frac{1}{2}\theta^2 + O(\theta^3)$.

By the hyperbolic law of cosines, we have $\cosh l = (\cosh 2)^2 + (\sinh 2)^2 \cos \alpha = \cosh 4 - (\sinh 2)^2(1 - \cos \alpha) = \cosh 4 + \frac{1}{2}(\sinh 2)^2 \alpha^2 + O(\alpha^3)$. Moreover, $\cosh l = \cosh(4 - h) = \cosh 4 - \sinh 4 \ h + O(h^2) = \cosh 4 - \frac{1}{2}\sinh 4 \ \theta^2 + O(\theta^3)$. Combining these equations we obtain that $\lim_{\theta \to 0} \alpha^2/\theta^2 = 2 \coth 2$ and the lemma follows. $\square$

We will frequently use the following application of the lemma. Let (X, x) be an Alexandrov space of curvature ≥ -1. Suppose that (a, b) is a 1-strainer at x of length ≤ 2 with comparison angle $\tilde{\angle}_x(a, b) \geq \pi - \theta$ with respect to some comparison curvature value $k \in [-1, 0]$; and hence in particular with *euclidean* comparison angle (with respect to $k = 0$) $\geq \pi - \theta$. Then for sufficiently small θ, i.e. $\theta \in (0, \theta_0)$ for some universal $\theta_0 > 0$, Lemma 3.2 implies that the strainer (a, b) has comparison angle $\geq \pi - \frac{3}{2}\theta$ with respect to the comparison curvature value -1.

For future reference, we also compute that $\alpha(\frac{\pi}{2}) < \frac{3\pi}{4}$ by solving the above equations with $l = 8^{\frac{1}{2}}$ for α. In other words, if we have an equilateral 1-strainer of length ≤ 2 which is $\frac{\pi}{2}$-straight with respect to some comparison curvature value in $[-1, 0]$, it is still $\frac{3\pi}{4}$-straight with respect to all other comparison curvature values in $[-1, 0]$.

3.2 Uniform local approximation by cones

Alexandrov spaces can in every point be arbitrarily well locally approximated by their tangent cone if one zooms in sufficiently far. We need a quantitative version of this infinitesmial cone-likeness, that is, we need uniform scales on which one can well approximate by cones. (Compare the scaling argument in [MT08, 3.5].)

Definition 3.3 (Local approximation by cones). We say that the Alexandrov ball $B(y, 1)$ is in the point $z \in B(y, \frac{1}{2})$ on the scale $s \leq \frac{1}{2}$ μ-well *approximated by a cone* if the rescaled

pointed ball $s^{-1} \cdot (B(z, s), z)$ has Gromov-Hausdorff distance $< \mu$ from the euclidean cone of radius 1 over some Alexandrov space with curvature ≥ 1 and with base point in the tip of the cone.

The base of the approximating cone may be empty, in which case the cone is just a point.

The following result says that Alexandrov balls of dimension $\leq d$ or, more generally, which are *roughly $\leq d$-dimensional* in the sense that $\bar\theta_{d\frac{1}{2}}$-straight $d\frac{1}{2}$-strainers must be very short, can be arbitrarily well locally approximated by cones of dimension $\leq d$.

Proposition 3.4 (Local approximation by cones on uniform scales). *For $d \in \mathbb{N}$ and $\sigma, \mu > 0$ exist scales $0 < s_{d\frac{1}{2}} = s_{d\frac{1}{2}}(\sigma, \mu) << s_1 = s_1(d, \sigma, \mu) << \sigma$ such that:*

An Alexandrov ball $B(y, 1)$ with curvature ≥ -1 and without $\bar\theta_{d\frac{1}{2}}$-straight $d\frac{1}{2}$-strainers of length $\geq s_{d\frac{1}{2}}$ can in every point $z \in B(y, \frac{1}{2})$ on some scale $s(z) \in [s_1, \sigma]$ be μ-well approximated by a cone of dimension $\leq d$.

Proof. Consider a sequence of Alexandrov balls $B(y_k, 1)$ with curvature ≥ -1 and without $\bar\theta_{d\frac{1}{2}}$-straight $d\frac{1}{2}$-strainers of length $\geq \frac{\sigma}{k}$, and suppose that there exist points $z_k \in B(y_k, \frac{1}{2})$ such that $B(y_k, 1)$ can in z_k not be μ-well approximated by a cone of dimension $\leq d$ on any scale $s \in [\frac{\sigma}{k}, \sigma]$. The $(B(y_k, 1), z_k)$ Gromov-Hausdorff converge to a pointed Alexandrov ball $(B(y_\infty, 1), z_\infty)$ with curvature ≥ -1 and dimension $\leq d$, and $z_\infty \in B(y_\infty, \frac{1}{2})$. Now $B(y_\infty, 1)$ can on a sufficiently small scale $s' < \sigma$ be μ-well approximated in z_∞ by (the truncation at radius 1 of) its tangent cone. Since $\frac{\sigma}{k} < s'$ for large k, we obtain a contradiction. $\qquad\square$

Throughout the paper we will fix some small value for σ, say $\sigma = \frac{1}{2010}$.

3.3 Islands without strainers

Building on 3.4 we will now divide our roughly $\leq d$-dimensional Alexandrov balls $B(y, 1)$ with curvature ≥ -1 into two regions according to the (non)existence of good 1-strainers on a uniform scale. We will show that the points without such strainers accumulate in "islands" which are uniformly separated from each other.

Definition 3.5 (Hump). For small $\theta > 0$ we call a point $z \in B(y, \frac{1}{2})$ a (θ, μ)-*hump*, if the base of the approximating cone provided by 3.4 has diameter $< \pi - \frac{\theta}{2}$.

Let $H = H_{\theta,\mu} \subset B(y, \frac{1}{2})$ denote the subset of (θ, μ)-humps, and let $S = S_{\theta,\mu} \subset B(y, 1)$ denote the subset of points admitting $< \theta$-straight 1-strainers of length $> \frac{1}{11} s_1(d, \sigma, \mu)$, cf. 3.4. We are interested in the distribution of the set $H - S$ for small θ and μ. (Our notation suppresses the dependence on d. Later we will only need the case $d = 2$.)

If the approximation accuracy μ is sufficiently small, then humps and non-humps have the following properties.

Lemma 3.6. *For sufficiently small $\theta > 0$ (i.e. $0 < \theta \leq \theta_0$ for some positive θ_0) there exists $\mu_0(\theta) > 0$ such that for $\mu \in (0, \mu_0(\theta)]$ holds:*

(i) A (θ, μ)-hump z admits no $\frac{\theta}{4}$-straight 1-strainers of length $\frac{1}{111} s(z)$, but

(ii) all points in the closed annulus $\overline{A}(z; \frac{1}{10}s(z), \frac{9}{10}s(z))$ do admit $\frac{\theta}{11}$-straight 1-strainers of length $> \frac{1}{11}s(z)$, i.e. $\overline{A}(z; \frac{1}{10}s(z), \frac{9}{10}s(z)) \subset S_{\theta,\mu}$.

(iii) Moreover, if $y' \in B(y, \frac{1}{2})$ is no (θ, μ)-hump, then it admits $< \theta$-straight 1-strainers of length $> \frac{99}{100}s(z) > \frac{1}{11}s_1(d, \sigma, \mu)$, i.e. $B(y, \frac{1}{2}) \subset H_{\theta,\mu} \cup S_{\theta,\mu}$.

Proof. Property (i) follows from the fact that euclidean comparison angles (with respect to comparison curvature value 0) are larger than their hyperbolic equivalents (with respect to comparison curvature value -1).

For the proof of (ii), every point in $\overline{A}(z; \frac{1}{10}s(z), \frac{9}{10}s(z))$ can be made arbitrarily close to the midpoint of a segment of length $> \frac{2}{11}s(z)$ if μ is chosen sufficiently small. This implies the existence of a $< \theta$-straight 1-strainer of length $> \frac{1}{11}s(z)$ at every point in $\overline{A}(z; \frac{1}{10}s(z), \frac{9}{10}s(z))$ for sufficiently small μ.

To prove (iii), we observe that for every $\theta > 0$ and sufficiently small $\mu > 0$ every point y' which is not a (θ, μ)-hump z admits a 1-strainer (a, b) of length $> \frac{1}{11}s(y')$ which is $\frac{2}{3}\theta$-straight as a *euclidean* 1-strainer, i.e. with respect to the comparison curvature value 0. By Lemma 3.2 and the discussion afterwards this strainer then is also $< \theta$-straight with respect to comparison curvature value -1 for sufficiently small $\theta > 0$. $\qquad\square$

Now we show that humps can be grouped into finitely many islands.

Proposition 3.7. *Let $d \in \mathbb{N}$ and $\sigma, \mu, \theta > 0$ such that $\mu \le \mu_0(\theta)$. Suppose that $B(y, 1)$ is an Alexandrov ball with curvature ≥ -1 and without $\bar{\theta}_{d\frac{1}{2}}$-straight $d\frac{1}{2}$-strainers of length $\ge s_{d\frac{1}{2}}(\sigma, \mu)$. Then there exist finitely many (θ, μ)-humps $z_j \in H_{\theta,\mu}$ such that*

$$B\left(y, \frac{1}{2}\right) \subset \left(\bigcup_j B(z_j, \frac{1}{10}s(z_j))\right) \cup S_{\theta,\mu}.$$

Moreover, $d(z_j, z_k) > \frac{9}{10}s(z_j)$ for $j \ne k$.

Proof. Let $z, z' \in H - S$. Then $z' \notin \overline{A}(z; \frac{1}{10}s(z), \frac{9}{10}s(z))$ and $z \notin \overline{A}(z'; \frac{1}{10}s(z'), \frac{9}{10}s(z'))$, i.e. $\frac{1}{s(z)}d(z, z'), \frac{1}{s(z')}d(z, z') \notin [\frac{1}{10}, \frac{9}{10}]$. If $z \in B(z', \frac{1}{10}s(z'))$ but $z' \notin B(z, \frac{1}{10}s(z))$, then $\frac{9}{10}s(z) < d(z, z') < \frac{1}{10}s(z')$, and so $B(z, \frac{1}{10}s(z)) \subset B(z', \frac{1}{9}s(z')) \subset B(z', \frac{1}{10}s(z')) \cup S$. There can be no infinite sequence of points $z, z', z'', \dots \in H - S$ such that $s(z) < \frac{1}{9}s(z') < \frac{1}{9^2}s(z'') < \dots$ because the scales take values in the bounded interval $[s_1, \sigma]$. Let $H_1 \subset H - S$ denote the subset of all points $z \in H - S$ for which no point $z' \in H - S$ exists with $\frac{9}{10}s(z) < d(z, z') < \frac{1}{10}s(z')$. We note that $\cup_{z \in H - S}B(z, \frac{1}{10}s(z)) \subset \cup_{z \in H_1}B(z, \frac{1}{10}s(z)) \cup S$.

By construction, the relation on H_1 defined by $z \sim z' :\Leftrightarrow z' \in B(z, \frac{1}{10}s(z))$ is reflexive and symmetric, and we have that $z \not\sim z' \Rightarrow d(z, z') > \frac{9}{10}s(z)$ and $> \frac{9}{10}s(z')$. To verify that the relation is also transitive, suppose that $z' \sim z \sim z''$ and $z' \not\sim z''$. Then $\frac{9}{20}(s(z') + s(z'')) < d(z', z'') \le d(z, z') + d(z, z'') < \frac{1}{10}(s(z') + s(z''))$, a contradiction. Thus, "$\sim$" is an equivalence relation on H_1. We call an equivalence class a (θ, μ)-*island*. Note that the island inhabited by z is contained in the intersection $\cap_{z' \sim z}B(z', \frac{1}{10}s(z'))$. Since inequivalent points $z', z'' \in H_1$ satisfy $d(z', z'') > \frac{9}{20}(s(z') + s(z''))$, islands are separated, $B(z', \frac{9}{20}s(z')) \cap B(z'', \frac{9}{20}s(z'')) = \emptyset$. In particular, there are only finitely many islands.

If $z \sim z'$ and $s(z') < 4s(z)$, then $B(z', \frac{1}{10}s(z')) \subset B(z, \frac{1}{2}s(z)) \subset B(z, \frac{1}{10}s(z)) \cup S$. Let $R \subset H_1$ be a subset which contains exactly one representative z from each island with almost maximal

scale value $s(z)$ (among its fellow islanders). Then $\cup_{z \in H_1} B(z, \frac{1}{10}s(z)) \subset \cup_{z \in R} B(z, \frac{1}{10}s(z)) \cup S$. R is finite and for any two distinct points $z', z'' \in R$ holds $d(z', z'') > \frac{9}{10}s(z')$. Altogether we obtain

$$H \subset \cup_{z \in H-S} B(z, \frac{1}{10}s(z)) \cup S \subset \cup_{z \in H_1} B(z, \frac{1}{10}s(z)) \cup S \subset \cup_{z \in R} B(z, \frac{1}{10}s(z)) \cup S$$

$\square$

3.4   The 1-strained region

We will now study the geometry of the region $S = S_{\theta, \mu}$ of points admitting good ($< \theta$-straight) 1-strainers which was introduced in section 3.3.

In the estimates provided in this section we will abstain from giving explicit constants although this could be done in each case. Instead we use the symbols $c, c', \dots$ to denote generic positive constants, i.e. constants which constantly change from estimate to estimate and which in each estimate (or assertion) take some fixed value independent of the other parameters $\theta, l, l', r, \lambda, \dots > 0$ involved. The estimates hold for sufficiently small values of the parameter θ, i.e. there exists some $\theta_0 > 0$ such that they hold for all $\theta \in (0, \theta_0]$. By decreasing the upper bound θ_0 for θ as we go along, we can also guarantee that the frequently occuring terms of the form $c\theta$ are as small as we wish. We will always assume that the upper bound θ_0 is sufficiently small such that the conclusions from Lemma 3.6 hold.

Throughout this section, let $X = B(x, 10)$ be an Alexandrov ball with curvature ≥ -1.

3.4.1   Local almost product structure

Let (a, b) be a $< 2\theta$-straight 1-strainer of length $> (1 - \theta)$ at x. We want to apply the following considerations to $< \theta$-straight 1-strainers on scale $\frac{1}{11} s_1(d, \sigma, \mu)$ which we rescale to length ≈ 1. They are then not necessarily anymore $< \theta$-straight with respect to comparison curvature value -1, but by Lemma 3.2 they still are $< 2\theta$-straight. In this way, we avoid that our constants depend on the scale $s_1(d, \sigma, \mu)$.

The estimates given below express that near x there is on a certain small scale an almost product structure with a one-dimensional factor in the direction of the strainer.

To begin with, the points near x admit 1-strainers close to (a, b) of comparable quality and almost the same length. More precisely, we have

$$\tilde{\angle}.(a, b) > \pi - c\theta \qquad \text{on } B(x, \theta) \tag{3.8}$$

with a certain constant $c > 1$. For future reference, let us denote by $C_0 > 1$ a constant such that (3.8) holds with $c = C_0$.

To verify (3.8), note that the function $d(a, \cdot) + d(b, \cdot)$ along xx' has first derivative $< \theta$ in x and second derivative $< c'$ (in a barrier sense) along the whole segment. Thus $d(a, x') + d(x', b) < d(a, x) + d(x, b) + c''\theta^2 < d(a, b) + c'''\theta^2$ which implies $\tilde{\angle}_{x'}(a, b) > \pi - c\theta$.

It follows that $f_{a,b} + d(b, \cdot) + const = -(f_{a,b} - d(a, \cdot)) + const = \frac{1}{2}(d(a, \cdot) + d(b, \cdot)) + const$ is $c\theta$-Lipschitz on $B(x, \theta)$.

One can define near x a coarse *flow* in the strainer direction which replaces the gradient flows of $d(a, \cdot)$ and $d(b, \cdot)$. For $t \in (-\theta, \theta)$ and $x' \in B(x, 3\theta)$ let $\Phi_t^{a,b}(x')$ be the intersection point $ax'b \cap f_{a,b}^{-1}(f_{a,b}(x') + t)$. The resulting maps $\Phi_t^{a,b} : B(x, 3\theta) \to X$ are well-defined only up to small ambiguity because the broken segments $ax'b$ need not be unique. Correspondingly, these maps are in general not continuous.

However, they are almost distance non-decreasing, $d(\Phi_t^{a,b}x_1, \Phi_t^{a,b}x_2) > (1 - c\theta)d(x_1, x_2) - c'\theta|t|$. This follows from triangle comparison applied to $\Delta(x_1, x_2, a)$ or $\Delta(x_1, x_2, b)$ and the fact that $|t| \le d(x_i, \Phi_t^{a,b}x_i) < (1 + c'\theta)|t|$ because $f_{a,b}$ has slope ≈ 1 along ax_ib. They are also almost inverse to each other, i.e. $\Phi_{-t}^{a,b}\Phi_t^{a,b}$ is $c\theta|t|$-close to the identity (where it is defined). To see this, consider the triangle $\Delta(x', \Phi_t^{a,b}x', \Phi_{-t}^{a,b}\Phi_t^{a,b}x')$ and note that $\tilde{\angle}_{\Phi_t^{a,b}x'}(x', \Phi_{-t}^{a,b}\Phi_t^{a,b}x') < c''\theta$ by (3.8) and again $|t| \le d(\Phi_t^{a,b}x', x'), d(\Phi_t^{a,b}x', \Phi_{-t}^{a,b}\Phi_t^{a,b}x') < (1 + c'\theta)|t|$. It follows that we have also an upper bound $d(\Phi_t^{a,b}x_1, \Phi_t^{a,b}x_2) \le (1 + c'''\theta)d(x_1, x_2) + c''''\theta|t|$, i.e. the $\Phi_t^{a,b}|_{B(x,\theta)}$ are $(1 + c'''\theta t, c''''\theta t)$-quasi-isometric.

Let $x_1, x_2 \in B(x, \theta)$. Since $\angle \ge \tilde{\angle}$ and geodesic triangles in $\Sigma_{x_i}X$ have circumference $\le 2\pi$, (3.8) yields $\angle_{x_1}(a, x_2) + \angle_{x_1}(b, x_2), \angle_{x_2}(a, x_1) + \angle_{x_2}(b, x_1) < \pi + c\theta$. Since also $\tilde{\angle}_{x_1}(a, x_2) + \tilde{\angle}_{x_2}(a, x_1), \tilde{\angle}_{x_1}(b, x_2) + \tilde{\angle}_{x_2}(b, x_1) > \pi - c'\theta$, we obtain that angles and comparison angles with a strainer direction almost coincide,

$$\angle_{x_1}(a, x_2) - \tilde{\angle}_{x_1}(a, x_2) < c\theta, \tag{3.9}$$

and

$$|\tilde{\angle}_{x_1}(a, x_2) + \tilde{\angle}_{x_1}(b, x_2) - \pi| < c\theta, \qquad |\tilde{\angle}_{x_1}(a, x_2) - \tilde{\angle}_{x_2}(b, x_1)| < c\theta. \tag{3.10}$$

As a consequence, $d(a, \cdot)$ is near x *almost affine* along segments in the sense that its slope is almost constant. More precisely,

$$d(a, x_1) - d(a, x_2) = d(x_1, x_2) \cos \alpha \tag{3.11}$$

with some angle α satisfying $|\alpha - \tilde{\angle}_{x_1}(a, x_2)| < c\theta$, as follows from (3.9,3.10) and the monotonicity of the cosine by integrating the derivative of $d(a, \cdot)$ along x_1x_2. Estimates of the same form hold for $d(b, \cdot)$ and $f_{a,b}$. (Note that (3.11) and the corresponding estimate $d(b, x_1) - d(b, x_2) = d(x_1, x_2) \cos \beta$ with $\beta = \pi - \alpha'$ satisfying $|\alpha' - \tilde{\angle}_{x_1}(a, x_2)| < c'\theta$ yield that $f_{a,b}(x_1) - f_{a,b}(x_2) = d(x_1, x_2) \cos \alpha''$ with $\cos \alpha'' = \frac{1}{2}(\cos \alpha + \cos \alpha')$, i.e. $|\alpha'' - \tilde{\angle}_{x_1}(a, x_2)| < c''\theta$.) It follows that there exists a constant $L > 0$ such that the function

$$f_{a,b} - \frac{f_{a,b}(x_2) - f_{a,b}(x_1)}{d(x_2, x_1)} d(x_1, \cdot) \tag{3.12}$$

and the analogous functions derived from $d(a, \cdot)$ and $d(b, \cdot)$ are $L\theta$-Lipschitz continuous along the segment x_1x_2.

In particular, the *cross sections* of a strainer are topological hypersurfaces *almost perpendicular* to it: If $x_1, x_2 \in f_{a,b}^{-1}(t) \cap B(x, \theta)$, then (3.9,3.10) imply

$$\frac{\pi}{2} - c\theta < \tilde{\angle}_{x_1}(a, x_2) \le \angle_{x_1}(a, x_2) < \frac{\pi}{2} + c\theta \tag{3.13}$$

because the comparison angles of the triangles $\Delta(x_1, x_2, a)$ amd $\Delta(x_1, x_2, b)$ almost agree. Moreover, the functions $d(a, \cdot)$, $d(b, \cdot)$ and $f_{a,b}$ are $L\theta$-Lipschitz continuous on any segment with endpoints in the same (piece of) cross section $f_{a,b}^{-1}(t) \cap B(x, \theta)$.

The maps $\Phi_t^{a,b}$ can be used to compare cross sections, since by their definition they satisfy $\Phi_t^{a,b}(f_{a,b}^{-1}(t') \cap B(x, 2\theta)) \subset f_{a,b}^{-1}(t' + t)$.

Regarding the *size* of cross sections of n-strainers $(a_1, b_1, \ldots, a_n, b_n)$ we obtain from (3.8,3.13): If points $x_1, x_2 \in f_{a_1,b_1,\ldots,a_n,b_n}^{-1}(t) \cap B(x, \theta)$ have distance l, then they admit $c\theta$-straight $n\frac{1}{2}$-strainers of length l.

Connectivity of cross sections. Let $x_1, x_2 \in f_{a,b}^{-1}(t) \cap B(x, \theta)$. For the midpoint m of (a segment) $x_1 x_2$ holds $|f_{a,b}(m) - t| < c\theta d(x_1, x_2)$. As above, by amb with $f_{a,b}^{-1}(t)$ we find an almost midpoint $y \in f_{a,b}^{-1}(t)$ at distance $< (\frac{1}{2} + c\theta) d(x_1, x_2)$ from x_1 and x_2. Iterating this procedure yields a continuous curve in $f_{a,b}^{-1}(t) \cap B(x, 3\theta)$ connecting x_1 and x_2. (Here one uses the earlier estimates with the parameter 3θ instead of θ.)

To simplify notation, let us put $\Sigma_{y;a,b}^o := \Sigma_{y;a,b} \cap B(x, \theta)$.

Lemma 3.14 (Projecting to cross sections). *Let $y, y_1 \in B(x, \theta)$ and let z_1 be the intersection point $ay_1b \cap \Sigma_{y;a,b}$. Then*

$$\left| d(y_1, z_1) - d(y, y_1) \cdot |\cos \tilde{\angle}_y(a, y_1)| \right| < c\theta d(y, y_1) \tag{3.15}$$

and

$$|d(y, z_1) - d(y, y_1) \sin \tilde{\angle}_y(a, y_1)| < c'\theta d(y, y_1). \tag{3.16}$$

In particular,

$$|d(y, z_1)^2 + d(z_1, y_1)^2 - d(y, y_1)^2| < c''\theta d(y, y_1)^2. \tag{3.17}$$

Proof. We put $l = d(y, y_1)$ and $\alpha_1 = \tilde{\angle}_y(a, y_1)$.

We have $|f_{a,b}(y_1) - f_{a,b}(y)| \leq d(y_1, z_1) \leq (1 + c\theta)|f_{a,b}(y_1) - f_{a,b}(y))|$ and, due to (3.11) and the remark thereafter, $f_{a,b}(y_1) - f_{a,b}(y) = -l \cos \alpha_1'$ with $|\alpha_1' - \alpha_1| < c\theta$. This yields (3.15).

To estimate $d(y, z_1)$, we consider a comparison triangle for $\Delta(y, y_1, z_1)$. In view of (3.10), we may exchange a and b, and therefore assume without loss of generality that $z_1 \in y_1 a$. Then $\alpha_1 > \frac{\pi}{2} - c\theta$, cf. (3.13). Regarding $\tilde{\angle}_{y_1}(z_1, y)$, we have $\tilde{\angle}_{y_1}(a, y) \leq \tilde{\angle}_{y_1}(z_1, y) \leq \angle_{y_1}(z_1, y) = \angle_{y_1}(a, y)$ and hence

$$|\tilde{\angle}_{y_1}(z_1, y) - (\pi - \alpha_1)| < c\theta$$

because of (3.9) and $\pi - c\theta < \tilde{\angle}_{y_1}(a, y) + \alpha_1 \leq \pi$. This information implies (3.16). (Whether we use a hyperbolic or euclidean comparison triangle to compute the length of the side $y_1'z'$ corresponding to y_1z, causes only a difference by a factor $< \frac{\sinh l}{l} < 1 + \theta^2 < 1 + c\theta$ (due to the distortion of the exponential map for hyperbolic plane up to radius l) and we may therefore work with a euclidean one. Then $|\frac{1}{l}d(y', z_1') - \sin \tilde{\angle}_{y_1}(z_1, y)| < |\cos(\pi - \alpha_1) - \cos \tilde{\angle}_{y_1}(z_1, y)| + c\theta < c'\theta$.)

Finally, (3.17) is a direct consequence. $\square$

3.5 The roughly ≤ 2-dimensional case

We assume now in addition that $B(x, 10)$ is *roughly ≤ 2-dimensional* in the sense that there are no $\bar{\theta}_{2\frac{1}{2}}$-straight $2\frac{1}{2}$-strainers of length λ for some (very small) $\lambda > 0$.

3.5.1 Cross sections of 1-strainers

We continue our discussion in section 3.4. The next two results express that the cross sections of good 1-strainers are now roughly $\leq$ 1-dimensional.

Lemma 3.18. *Let (a,b) be a $C_0\theta$-straight $> (1-\theta^{\frac{1}{2}})$-long 1-strainer at a point $y \in B(x,5)$. Let y_1y_2 be a segment of length l with endpoints in $B(y,\frac{\theta}{3})$ and with midpoint m. Let y' be a point with $d(m,y') > \theta l$. Suppose that $f_{a,b}$ is $3L\theta$-Lipschitz on the segments y_1y_2 and $my' \cap B(m,\theta l)$. Then $\angle_m(y_i, y') < c\theta$ for $i = 1$ or 2, if $\lambda = \lambda(\theta, l)$ is sufficiently small.*

If also $d(y_1, y') \leq \frac{l}{3}$, then $\angle_m(y_1, y') < c\theta$.

Proof: Let $z' \in my'$ be the point at distance θl from m. Since f_{y_1,y_2} has slope $\equiv 1$ on y_1y_2, there exists a point $z \in y_1y_2 \cap \overline{B}(m,\theta l)$ with $f_{y_1,y_2}(z) = f_{y_1,y_2}(z')$. It satisfies

$$|f_{a,b}(z) - f_{a,b}(z')| \leq 6L\theta^2 l \qquad (3.19)$$

because $f_{a,b}$ is $3L\theta$-Lipschitz on zm and mz'.

The Lipschitz assumption also implies that $|\angle.(y_i,a) - \frac{\pi}{2}|, |\angle.(y_i,b) - \frac{\pi}{2}| < c\theta$ on (the interior of) y_1y_2 and hence $|\tilde{\angle}.(y_i,a) - \frac{\pi}{2}|, |\tilde{\angle}.(y_i,b) - \frac{\pi}{2}| < c'\theta$ by (3.9). Thus the quadrupel (a,b,y_1,y_2) is a $c'\theta$-straight 2-strainer at z.

The $1\frac{1}{2}$-strainer (y_1, y_2, z') at z is $c\theta$-straight by (3.13).

We consider now the $1\frac{1}{2}$-strainer (a, b, z') at z and estimate $\tilde{\angle}_z(a, z')$. Since $d(z,z') \leq 2\theta l$ and $f_{a,b} - d(a,\cdot)$ is $c\theta$-Lipschitz on $B(y,\theta)$, (3.19) translates to $|d(a,z) - d(a,z')| < c'\theta^2 l$. With (3.11) follows $d(z,z')|\cos\tilde{\angle}_z(a,z')| < c'\theta^2 l + c''\theta d(z,z') < c'''\theta^2 l$.

If $d(z,z') > 2c'''\bar{\theta}_{2\frac{1}{2}}^{-1}\theta^2 l$ (with the constant c''' from the last estimate), then $|\cos\tilde{\angle}_z(a,z')| < \frac{1}{2}\bar{\theta}_{2\frac{1}{2}}$ and $|\tilde{\angle}_z(a,z') - \frac{\pi}{2}| < \bar{\theta}_{2\frac{1}{2}}$. Similarly, $|\tilde{\angle}_z(b,z') - \frac{\pi}{2}| < \bar{\theta}_{2\frac{1}{2}}$, and it follows that (a,b,y_1,y_2,z') is a $< \bar{\theta}_{2\frac{1}{2}}$-straight $2\frac{1}{2}$-strainer at z with length $> 2c'''\bar{\theta}_{2\frac{1}{2}}^{-1}\theta^2 l$. This is a contradiction if $\lambda = \lambda(\theta, l)$ is sufficiently small, and we conclude that $d(z,z') < c''''\theta^2 l$ and consequently $\tilde{\angle}_m(z,z') < c\theta$.

Suppose without loss of generality that $z \in my_1$. Then $\angle_m(z,z') - \tilde{\angle}_m(z,z') \leq \angle_m(y_1,z') - \tilde{\angle}_m(y_1,z') < c'\theta$, where the last inequality follows from (3.9) after rescaling by the factor l^{-1}. Thus $\angle_m(y_1,y') = \angle_m(z,z') < c''\theta$, which shows the first assertion.

For the second assertion, suppose that $d(y_1,y') \leq \frac{l}{3}$ but $\angle_m(y_2,y') < c\theta$. Then y' is close to my_2 or y_2 is close to my'. Since $d(y_2,y') \geq \frac{2l}{3}$, only the second alternative can occur and $d(m,y') \gtrsim \frac{l}{2} + \frac{2l}{3} > l$. On the other hand $d(m,y') \leq d(m,y_1) + d(y_1,y') < l$, a contradiction. Thus $\angle_m(y_1,y') < c\theta$. $\qquad\square$

Lemma 3.20 (Roughly one-dimensional cross section). *Let $l' \leq l \leq \frac{\theta}{50}$, $y \in B(x,5)$ and let (a,b) be a $C_0\theta$-straight $> (1 - \theta^{\frac{1}{2}})$-long 1-strainer at y. Suppose that $\mathrm{diam}(\Sigma^o_{y;a,b}) > 40l$.*

(i) Then there exists a point $y' \in \Sigma_{y;a,b} \cap B(y,l)$ and a segment $y'm$ of length $\geq 10l$, such that $f_{a,b}$ is $3L\theta$-Lipschitz on $y'm$ and $\Sigma_{y;a,b} \cap B(y,l) \subset N_{c\theta l}(y'm)$, if $\lambda = \lambda(\theta, l)$ is sufficiently small.

(ii) Moreover, if $\mathrm{diam}(\Sigma_{y;a,b} \cap B(y,l)) < \frac{199}{100}l$, then y' can be chosen in $B(y, \frac{199}{200}l)$ so that $\Sigma_{y;a,b} \cap B(y',r) \subset N_{c\theta r}(y'm)$ for all $r \in [l', l]$, if $\lambda = \lambda(\theta, l')$ is sufficiently small.

Proof. (i) By assumption, there exists $q \in \Sigma_{y;a,b}$ with $d(y,q) = 20l$. (Recall from section 3.4.1 that $\Sigma_{y;a,b}$ is near y path connected.) Let m be the midpoint of yq. Since $f_{a,b}$ is $L\theta$-Lipschitz on yq, we have that $|f_{a,b}(m) - f_{a,b}(y)| < 10L\theta l$. Thus for every point $z \in \Sigma_{y;a,b} \cap B(y,l)$ the function $f_{a,b}$ has along the segment mz Lipschitz constant $< \frac{10}{9}L\theta + L\theta = \frac{19}{9}L\theta < 3L\theta$, cf. (3.12), and 3.18 yields that $\angle_m(y,z) < c\theta$. It follows that the segments mz have pairwise angles $< 2c\theta$ and are all contained in the $c'\theta l$-neighborhood of an almost longest one among them. We choose y' as its endpoint.

(ii) By part (i), the piece of cross section $\Sigma_{y;a,b} \cap B(y,l)$ is $c\theta l$-close to a subsegment $y'z \subset y'm$ where $z \in y'm$ is a point with $d(y,z) = l$ and $|d(z,m) + l - d(y,m)| < c'\theta l$. Hence $d(y,y') < (\frac{99}{100} + c''\theta)l$. The points in $\Sigma_{y;a,b} \cap B(y,l)$, which are further away from m than y', must be $2c\theta l$-close to y', i.e. $d(m,\cdot)|_{\Sigma_{y;a,b}\cap B(y,l)}$ assumes a maximum in $\Sigma_{y;a,b} \cap \overline{B}(y', 2c\theta l)$. We replace y' by this maximum and then have $d(y,y') < \frac{199}{200}l$.

Also by (i), $\Sigma_{y;a,b}$ contains no $\frac{l}{1000}$-long $\frac{\pi}{2}$-straight 1-strainer at y'.

Suppose that $\Sigma_{y;a,b}$ contains a $\theta l'$-long $\frac{\pi}{2}$-straight 1-strainer at y'. Using (i) at the point y' and on the scale $\theta l'$, it follows that for sufficiently small $\lambda = \lambda(\theta, l')$ there exists a segment τ of length almost $2\theta l'$, say, of length $\frac{199}{100}\theta l'$ along which $f_{a,b}$ is $c\theta$-Lipschitz and such that y' lies at distance $< c'\theta^2 l'$ from the midpoint of τ. (When applying part (i) to points nearby y, the 1-strainer (a,b) may only be $c\theta$-straight for some constant $c > C_0$ and $(1 - 2\theta^{\frac{1}{2}})$-long at these points, and we use a version of part (i) with appropriate different constants.)

We consider $d(m,\cdot)$ along the middle third τ' of τ. The function $f_{a,b}$ is $c\theta$-Lipschitz along τ and along all segments zm initiating in interior points z of τ'. Lemma 3.18 implies for sufficiently small $\lambda = \lambda(\theta, l')$ that these segments zm have angles $< c'\theta$ with τ'. This means that $d(m,\cdot)$ has slope $\approx \pm 1$ along τ', i.e. the directional derivatives of $d(m,\cdot)$ in directions tangent to τ' take values in $[-1, -1 + c''\theta^2) \cup (1 - c''\theta^2, 1]$. (See e.g. [BGP92, §11] for a discussion of directional derivatives of distance functions.) If at some interior point z_0 of τ' the directional derivatives in the two antipodal directions tangent to τ' are both negative, then $d(m,\cdot)$ decays with slope ≈ -1 along both subsegments of τ' with initial point z_0. In particular, z_0 is a maximum of $d(m,\cdot)|_{\tau'}$. If such a point z_0 does not exist, then $d(m,\cdot)|_{\tau'}$ is almost affine, i.e. with respect to an appropriate orientation of τ' it increases with slope ≈ 1 along the whole segment.

Since y' is a maximum of $d(m,\cdot)|_{\Sigma_{y;a,b}\cap B(y', \frac{l}{200})}$, it follows that $d(m,\cdot)|_{\tau'}$ attains a maximum at a point $y'' \in \tau'$ close to the midpoint of τ', more precisely, at distance $< c''\theta^2 l'$ from y'. Furthermore, there exist at least two segments σ_1 and σ_2 connecting y'' to m whose initial directions $\dot\sigma_i(0)$ (with respect to unit speed parametrizations starting at y'') are close to the two antipodal directions of τ' at y''. Each endpoint of τ lies at distance $< c'''\theta^2 l'$ from one of the two segments σ_1 and σ_2, and hence $d(\sigma_1(\theta l'), \sigma_2(\theta l')) > \frac{19}{10}\theta l'$.

Applying part (i) at y'' on the scales between $\theta l'$ and l yields that for $\lambda(\theta, l')$ sufficiently small the continuous function $t \mapsto \frac{1}{t}d(\sigma_1(t), \sigma_2(t))$ on $[\theta l', l]$ takes values close to 0 and 2, i.e. in $[0, c\theta) \cup (2 - c\theta, 2]$. However, by the above, it has value ≈ 0 for $t = \frac{l}{1000}$ and value ≈ 2 for $t = \theta l'$, a contradiction. (Note that the smaller the scale, the smaller λ has to be, and there exists a λ which serves simultaneously for all scales $s \in [\theta l', l]$.)

Thus $\Sigma_{y;a,b}$ contains no $\theta l'$-long $\frac{\pi}{2}$-straight 1-strainer at y'. It follows, again by part (i) on the scale $\theta l'$, that $\text{diam}(\Sigma_{y;a,b} \cap \partial B(y', \theta l')) < c\theta^2 l'$, and hence every point $z \in \Sigma_{y;a,b}$ at distance

$d \in [\theta l', l]$ from y' has distance $< c\theta d$ from $y'm$. This proves the assertion. $\qquad \square$

Lemma 3.21 (Spreading 1-strainers). *Let l, l', y and (a, b) be as in 3.20. Let $z, z_1, z_2 \in \Sigma_{y;a,b} \cap B(y, l)$ such that (z_1, z_2) is a $\frac{\pi}{2}$-straight 1-strainer of length $r \in [l', l]$ at z. Then there exists a constant $C_1 \geq C_0$ such that for sufficiently small $\lambda = \lambda(\theta, l')$ holds:*

(i) The 1-strainer (z_1, z_2) at z is $< C_1\theta$-straight.

(ii) All points $u \in azb \cap B(z, \min(100\theta^{-\frac{3}{4}}r, \theta))$ admit $< C_1\theta$-straight $\frac{r}{2}$-long 1-strainers contained in $\Sigma_{u;a,b}$, and hence such a 2-strainer contained in $\Sigma_{u;a,b} \cup aub$).

Proof. (i) This is a consequence of 3.20(i) applied at the point z on the scale r.

(ii) Now we use the $(1 + c\theta, c'\theta t)$-quasi-isometry property of the maps $\Phi_t^{a,b}$ up to distance $\approx \theta$ from y. We may assume that $u = \Phi_t^{a,b}z$ with $|t| < \min(100\theta^{-\frac{3}{4}}r, \theta)$ and put $u_i = \Phi_t^{a,b}z_i$. Then $d(u, u_i) < (1 + c\theta)r + c'\theta t$ and $d(u_1, u_2) > (1 - c\theta)\frac{99}{50}r - c'\theta t$, using that $d(z_1, z_2) \approx 2r$ according to part (i). We obtain a $\frac{\pi}{3}$-straight 1-strainer (u_1, u_2) at u contained in $\Sigma_{u;a,b}$ and with length $\approx r$. Close to the midpoints of the segments uu_i we find a $\frac{\pi}{2}$-straight 1-strainer (u_1', u_2') at u contained in $\Sigma_{u;a,b}$ and with length $\frac{r}{2}$. (Compare the argument for the connectivity of cross sections.) Applying part (i) again on the scale $\frac{r}{2}$ yields the $c\theta$-straight $\frac{r}{2}$-long 1-strainer, once we can make sure that $\operatorname{diam}(\Sigma_{u;a,b}^o) > 20r$. But this follows from our assumption that $\operatorname{diam}(\Sigma_{y;a,b}^o) > 40l \geq 40r$ by using the maps $\Phi_t^{a,b}$ as before. $\qquad \square$

Remark 3.22. As in part (i) of the lemma, we obtain that if the 1-strainer (z_1, z_2) is $\frac{3\pi}{4}$-straight, it is also $< C_1\theta$-straight and hence in particular $\frac{\pi}{2}$-straight.

3.5.2 Edges

In view of the local product structure, the points y' obtained in 3.20(ii) can be considered as points "near the edge" of our space. They are characterized by the property that they admit no long 1-strainers contained in the cross section. We will now investigate the geometry near the edge, keeping the assumption of rough 2-dimensionality from section 3.5.

Good 1-strainers at points near the edge must be almost perpendicular to the cross section if they are not too short:

Lemma 3.23 (Almost unique 1-strainers near the edge). *Let l, y and (a, b) be as in 3.20. Suppose that (a', b') is a $c\theta^{\frac{1}{2}}$-straight 1-strainer of length $\geq l$ at y.*

(i) If $\Sigma_{y;a,b}$ contains no $\frac{\pi}{2}$-straight $c'l$-long 1-strainer at y, then $\angle_y(a, a'), \angle_y(a, b'), \angle_y(b, a'), \angle_y(b, b') \notin [\frac{\pi}{100}, \frac{99}{100}\pi]$, if $\lambda = \lambda(\theta, l)$ is sufficiently small.

(ii) If $\Sigma_{y;a,b}$ contains no $\frac{\pi}{2}$-straight θl-long 1-strainer at y, then $\angle_y(a, a'), \angle_y(a, b'), \angle_y(b, a'), \angle_y(b, b') \notin [c'\theta^{\frac{1}{2}}, \pi - c'\theta^{\frac{1}{2}}]$, if $\lambda = \lambda(\theta, l)$ is sufficiently small.

In both cases, y admits no $c\theta^{\frac{1}{2}}$-straight l-long 2-strainer.

Proof. There exists a $c\theta^{\frac{1}{2}}$-straight l-long 1-strainer (y_1, y_2) at y. (Choose $y_1 \in ya'$ and $y_2 \in yb'$.) We put $\alpha_i = \tilde{\angle}_y(a, y_i)$.

By 3.11 and the remark afterwards, $|f_{a,b}(y_i) - f_{a,b}(y) + l\cos\alpha_i| < c\theta l$. Thus for $u_1 = ay_1b \cap \Sigma_{y_2;a,b}$ holds $(1 - c\theta)d(y_1, u_1) < |f_{a,b}(y_1) - f_{a,b}(y_2)| \leq |f_{a,b}(y_1) - f_{a,b}(y)| + |f_{a,b}(y) - f_{a,b}(y_2)|$

and
$$\frac{1}{l}d(y_1, u_1) < |\cos\alpha_1| + |\cos\alpha_2| + c\theta,$$
compare the proof of 3.14.

Let $z_i = ay_ib \cap \Sigma_{y;a,b}$. According to 3.14, we have $|d(y, z_i) - l\sin\alpha_i| < c\theta l$. Now the rough one-dimensionality of the cross section $\Sigma_{y;a,b}$, i.e. 3.20(i) applied on the scale l, and our assumption in part (ii) yield that $d(y, y') < c'\theta l$ and $|d(z_1, z_2) - |d(y, z_1) - d(y, z_2)|| < c''\theta l$. Hence $|\frac{1}{l}d(z_1, z_2) - |\sin\alpha_1 - \sin\alpha_2|| < c'''\theta$. With the metric properties of the maps $\Phi_t^{a,b}$ follows

$$\left|\frac{1}{l}d(u_1, y_2) - |\sin\alpha_1 - \sin\alpha_2|\right| < c\theta.$$

Since (y_1, y_2) is $c\theta^{\frac{1}{2}}$-straight, we have $d(y_1, y_2) > (2 - c\theta)l$, and (3.17) implies

$$(|\cos\alpha_1| + |\cos\alpha_2|)^2 + (\sin\alpha_1 - \sin\alpha_2)^2 > 4 - c'\theta.$$

Writing $\alpha_i = \frac{\pi}{2} + \beta_i$, the last inequality becomes $4\sin^2\frac{|\beta_1|+|\beta_2|}{2} = (\sin|\beta_1| + \sin|\beta_2|)^2 + (\cos\beta_1 - \cos\beta_2)^2 > 4 - c'\theta$ and

$$\sin\frac{|\beta_1| + |\beta_2|}{2} > 1 - c\theta.$$

Thus $|\beta_i| > \frac{\pi}{2} - c'\theta^{\frac{1}{2}}$ and $\tilde{\angle}_y(a, y_1), \tilde{\angle}_y(a, y_2), \tilde{\angle}_y(b, y_1), \tilde{\angle}_y(b, y_2) \notin [c'\theta^{\frac{1}{2}}, \pi - c'\theta^{\frac{1}{2}}]$.

To pass from comparison angles to angles, we note that if e.g. $\tilde{\angle}_y(a, y_1) < c\theta^{\frac{1}{2}}$, then $\angle_y(a, b') = \angle_y(a, y_2) \geq \tilde{\angle}_y(a, y_2) > \pi - c'\theta^{\frac{1}{2}}$. This shows (ii).

Under the assumption of part (i) we obtain weaker but still useful estimates. Now $|d(z_1, z_2) - |d(y, z_1) - d(y, z_2)|| < (2c' + c''\theta)l < 3c'l$ and it follows that $|\frac{1}{l}d(u_1, y_2) - |\sin\alpha_1 - \sin\alpha_2|| < c''$ and $|\beta_i| > \frac{\pi}{2} - c'''$. The constant c''' depends on the constant c' in the hypothesis of (i) and can be made arbitrarily small by choosing c' small enough. Assertion (i) follows. $\square$

Remark 3.24. Similarly, one shows e.g. that if $\Sigma_{y;a,b}$ contains no $\frac{\pi}{2}$-straight $\theta^{\frac{1}{2}}l$-long 1-strainer at y, then $\angle_y(a, a'), \angle_y(a, b'), \angle_y(b, a'), \angle_y(b, b') \notin [c'\theta^{\frac{1}{4}}, \pi - c'\theta^{\frac{1}{4}}]$, if $\lambda = \lambda(\theta, l)$ is sufficiently small.

We make the following choice of scales to quantify rough edges.

Definition 3.25 (Edgy points). A point $y \in B(x, 5)$ is called θ-*edgy* relative to a $< 2\theta$-straight 1-strainer (a, b) of length $> (1 - \theta)$ at y if $\operatorname{diam}(\Sigma^o_{y;a,b}) > \theta^{\frac{5}{2}}$ and $\Sigma_{y;a,b}$ contains no $\frac{\pi}{2}$-straight θ^4-long 1-strainer at y.

We say that $y \in B(x, 5)$ is θ-*weakly edgy* relative to a $< 2\theta$-straight 1-strainer (a, b) of length $> (1 - \theta)$ at y if $\operatorname{diam}(\Sigma^o_{y;a,b}) > \frac{1}{2}\theta^{\frac{5}{2}}$ and $\Sigma_{y;a,b}$ contains no $\frac{\pi}{2}$-straight $2\theta^4$-long 1-strainer at y.

We call $y \in B(x, 5)$ θ-*strongly edgy* relative to a $< 2\theta$-straight 1-strainer (a, b) of length $> (1 - \theta)$ at y if $\operatorname{diam}(\Sigma^o_{y;a,b}) > \frac{4}{3}\theta^{\frac{5}{2}}$ and $\Sigma_{y;a,b}$ contains no $\frac{\pi}{2}$-straight $\frac{3}{4}\theta^4$-long 1-strainer at y.

Lemma 3.26 (Points without 2-strainers are close to edgy). *Suppose that every point $y \in B(x, \theta)$ admits a $< 2\theta$-straight 1-strainer (a_y, b_y) of length $> (1 - \theta)$ and that (a, b) is a $< 2\theta$-straight 1-strainer of length $> (1 - \theta)$ at x such that $\Sigma^o_{x;a,b}$ has diameter $\geq 2\theta^{\frac{5}{2}}$ and contains no $< C_1\theta$-straight 2-strainer of length θ^4 centered at x.*

Then there is a point $z \in B(x, 2\theta^4)$ which is θ-strongly edgy relative to (a_z, b_z).

Proof. Applying 3.20 on the scale $l = l' = 2\theta^4$, we obtain that there is a point $z \in \Sigma_{x;a,b} \cap B(x, 2\theta^4)$ such that $\Sigma^o_{z;a,b} \subset \Sigma^o_{x;a,b}$ has diameter $\geq 2\theta^{\frac{5}{2}}$ and admits no $\frac{\pi}{2}$-straight 1-strainer of length $\frac{1}{2}\theta^4$. The 1-strainer (a, b) is still $< C_0\theta$-straight at z.

By assumption, the point z admits a $< 2\theta$-straight 1-strainer (a', b') of length $> (1 - \theta)$ By Lemma 3.23, the angles between the two 1-strainers (a, b) and (a', b') at z are (after changing the order of the strainer points if necessary) less than $c'\theta^{\frac{1}{2}}$.

For any point $z' \in \Sigma_{z;a',b'}$ we have $|\angle_z(a', z') - \frac{\pi}{2}| \leq c\theta$ (by 3.13), $|\angle_z(a', z') - \angle_z(a, z')| \leq c'\theta^{\frac{1}{2}}$ and $|\angle_z(a, z') - \tilde{\angle}_z(a, z')| \leq c''\theta$ (by 3.9). This implies that $|\tilde{\angle}_z(a, z') - \frac{\pi}{2}| \leq c'''\theta^{\frac{1}{2}}$. We now apply 3.14 and project the cross section $\Sigma^o_{z;a',b'}$ to $\Sigma_{z;a,b}$. This implies that $\Sigma_{z;a',b'}$ contains no $\frac{\pi}{2}$-straight 1-strainer of length $\frac{3}{4}\theta^4$ centered at z, since such a strainer would project to a 1-strainer of length $\geq \frac{1}{2}\theta^4$ in $\Sigma_{z;a,b}$ which must be $\frac{\pi}{2}$-straight by Lemma 3.21.

Similarly, we apply 3.14 to project $\Sigma^o_{z;a,b}$ to $\Sigma_{z;a',b'}$. This implies that $\operatorname{diam}\Sigma^o_{z;a',b'} \geq \frac{4}{3}\theta^{\frac{5}{2}}$. Thus, z is indeed θ-strongly edgy relative to (a', b'). $\qquad\square$

By 3.23(ii), at a θ-edgy point y, there exist no θ^3-long $c\theta^{\frac{1}{2}}$-straight 2-strainers and any two θ^3-long $c\theta^{\frac{1}{2}}$-straight 1-strainers have angle $< c'\theta^{\frac{1}{2}}$ (in the sense that their pairs of directions are $c'\theta^{\frac{1}{2}}$-Hausdorff close subsets in $\Sigma_y X$).

The almost uniqueness of 1-strainers at edgy points extends to uniform neighborhoods:

Lemma 3.27 (Almost uniqueness of 1-strainers extends). *Let y be θ-edgy relative to (a, b). Suppose that (a_i, b_i) are $C_0\theta$-straight 1-strainers of lengths $\in (1 - \theta^{\frac{1}{2}}, 1 + \theta^{\frac{1}{2}})$ at points $z_i \in B(y, \theta)$ for $i = 1, 2$. Then $\angle.(a_1, a_2), \angle.(a_1, b_2), \angle.(b_1, a_2), \angle.(b_1, b_2) \notin [c'\theta^{\frac{1}{2}}, \pi - c'\theta^{\frac{1}{2}}]$ on $B(y, \theta)$, if $\lambda = \lambda(\theta)$ is sufficiently small.*

Proof. The strainers (a_i, b_i) are $c'\theta$-straight at y, cf. (3.8), and hence $c\theta^{\frac{1}{2}}$-straight with the constant c as in the hypothesis of 3.23, because $c'\theta < c\theta^{\frac{1}{2}}$. Applying 3.23(ii) with $l = \theta^3$ in the edgy point y yields up to switching a_1 and b_1 that $\angle_y(a_1, a_2), \angle_y(b_1, b_2) < c''\theta^{\frac{1}{2}}$. Due to our condition on the lengths of the 1-strainers (a_i, b_i) it follows that they are $c'''\theta^{\frac{1}{2}}$-Hausdorff close (as two point subsets), which in turn implies that $\angle.(a_1, a_2), \angle.(b_1, b_2) < c''''\theta^{\frac{1}{2}}$ on $B(y, \theta)$. $\qquad\square$

Lemma 3.28 (Relative position of nearby edgy points). *Let x be θ-edgy relative to (a, b).*

(i) If $\lambda = \lambda(\theta)$ is sufficiently small, then all points in $B(x, \theta^3)$ which are θ-weakly edgy are contained in the $\theta^{\frac{15}{4}}$-neighborhood of axb.

(ii) Suppose that every point $y \in B(x, \theta)$ admits a $< 2\theta$-straight 1-strainer (a_y, b_y) of length $> (1 - \theta)$. Then for every point $y' \in axb \cap B(x, \theta^3)$ we have $\operatorname{diam}\Sigma_{y';a_{y'},b_{y'}} < \frac{3}{4}\theta^2$ or y' lies at distance $< \theta^{\frac{15}{4}}$ from a point z which is θ-strongly edgy relative to (a_z, b_z).

Proof. (i) Let $y \in \Sigma_{x;a,b} \cap A(x, \frac{1}{2}\theta^{\frac{15}{4}}, 2\theta^3)$. Then $\Sigma_{x;a,b}$ contains a $c\theta$-straight $\frac{1}{3}\theta^{\frac{15}{4}}$-long 1-strainer at y_1, cf. 3.20(i). By 3.21(ii), every point $u \in ayb \cap B(y, 2\theta^3)$ admits a $c\theta$-straight $\frac{1}{6}\theta^{\frac{15}{4}}$-long 1-strainer contained in $\Sigma_{u;a,b}$. By the metric properties of the maps $\Phi_t^{a,b}$, every point $z \in B(y, \theta^3)$ outside the $\theta^{\frac{15}{4}}$-neighborhood of axb lies at distance $< c'\theta^4$ from such a point u (for some such y) and therefore admits a $c''\theta$-straight $> \frac{1}{7}\theta^{\frac{15}{4}}$-long 1-strainer contained in $\Sigma^o_{z;a,b}$.

Suppose that z is θ-weakly edgy with respect to some 1-strainer (a', b'). Then by Lemma 3.27 the two 1-strainers (a, b) and (a', b') have angles $\leq c'\theta^{\frac{1}{2}}$ at z, and we can project $\Sigma^o_{z;a,b}$ to $\Sigma_{z;a',b'}$ as in the proof of Lemma 3.26 to obtain a contradiction.

(ii) Let $y' \in axb \cap B(x, \theta^3)$. Suppose that $\Sigma^o_{y';a,b}$ contains a $\frac{\pi}{2}$-straight $\frac{1}{2}\theta^{\frac{15}{4}}$-long 1-strainer at y. Then $y' = ayb \cap \Sigma_{x;a,b}$ has distance $< c\theta^4$ from x. By 3.21, $\Sigma^o_{x;a,b}$ then contains a $c'\theta$-straight $\frac{1}{4}\theta^{\frac{15}{4}}$-long 1-strainer at y', which is also a $\frac{\pi}{2}$-straight $> \frac{1}{5}\theta^{\frac{15}{4}}$-long 1-strainer at x. This contradicts the θ-edgyness of x. Thus $\Sigma^o_{y';a,b}$ contains no $\frac{1}{2}\theta^{\frac{15}{4}}$-long $\frac{\pi}{2}$-straight 1-strainer at y'.

If $\operatorname{diam}\Sigma^o_{y';a,b} < \theta^{\frac{9}{8}}$, by assumption there is a $< 2\theta$-straight 1-strainer (a', b') of length $> (1 - \theta)$ at y'. Projecting $\Sigma_{y';a,b}$ to $\Sigma_{y';a',b'}$ yields that $\operatorname{diam}\Sigma^o_{y';a',b'} < \frac{3}{4}\theta^2$.

Otherwise, 3.20(ii) applied on the scale $l = \theta^{\frac{15}{4}}$ yields that $\Sigma_{y';a',b'} \cap B(y, 2\theta^{\frac{15}{4}})$ contains a point z such that $\Sigma^o_{z;a',b'}$ has diameter $\geq \theta^{\frac{9}{4}}$ and admits no $\frac{\pi}{2}$-straight 1-strainer of length $\frac{1}{2}\theta^4$ centered at z. This implies that z is θ-strongly edgy relative to some 1-strainer (a_z, b_z) as in the proof of Lemma 3.26. $\qquad\square$

Lemma 3.29 (Almost parallel cross sections of edges). *Let x be a θ-edgy point relative to a θ-straight 1-strainer (a, b) with length $> (1 - \theta)$, and let y be θ-weakly edgy relative to another 1-strainer (a', b'). We consider the truncated cross sections $\check{\Sigma}_x := \Sigma_{x;a,b} \cap B(x, \frac{1}{2}\theta^3)$ and $\check{\Sigma}_y := \Sigma_{y;a',b'} \cap B(y, \frac{1}{2}\theta^3)$.*

(i) Suppose that $\check{\Sigma}_y$ and $\check{\Sigma}_y$ intersect. Then $d(x, y) < c\theta^{\frac{7}{2}}$, if $\lambda = \lambda(\theta)$ is sufficiently small.

(ii) Suppose that $d(x, y) \leq c\theta^{\frac{10}{3}}$. Then the Hausdorff distance $d_H(\check{\Sigma}_x, \check{\Sigma}_y)$ is less than $\theta^{\frac{99}{30}}$.

Proof. (i) Let z be one of the intersection points of the cross sections $\check{\Sigma}_x$ and $\check{\Sigma}_y$. By 3.27, $f_{a,b} - f_{a',b'}$ is $c\theta^{\frac{1}{2}}$-Lipschitz on $B(z, \frac{1}{2}\theta^3)$ (in fact, on $B(z, \theta)$), and hence $f_{a,b}$ is $c\theta^{\frac{1}{2}}$-Lipschitz on $\Sigma_{y;a',b'} \cap B(z, \frac{1}{2}\theta^3)$. It follows that $|f_{a,b}(x) - f_{a,b}(y)| = |f_{a,b}(z) - f_{a,b}(y)| < c\theta^{\frac{7}{2}}$. Since y is contained in the $\theta^{\frac{15}{4}}$-neighborhood of axb due to 3.28, we obtain that $d(y_1, y_2) < c'\theta^{\frac{7}{2}}$.

(ii) Let $z = ayb \cap \Sigma_{x;a,b}$ and consider a point $u \in \check{\Sigma}_y$. By 3.27, we have $\angle_y(a, a') \leq c'\theta^{\frac{1}{2}}$. Thus, we can apply 3.14 to project $\check{\Sigma}_y$ to $\Sigma_{y;a,b}$. In particular, the point u projects to a point v with $d(u, v) \leq c''\theta^{\frac{7}{2}}$ and $|d(y, v) - d(y, u)| \leq c'''\theta^3$.

Next, we apply the coarse flow $\Phi^{a,b}_{-f_{a,b}(y)}$ to transport $\Sigma_{y;a,b}$ to $\Sigma_{x;a,b}$. We have $z = \Phi^{a,b}_{-f_{a,b}(y)}$ and set $w := \Phi^{a,b}_{-f_{a,b}(y)}(v)$. Our condition that $d(x, y) \leq c\theta^{\frac{10}{3}}$ and the metric properties of the flow yield that $d(v, w) \leq c''''\theta^{\frac{10}{3}}$ and $|d(w, z) - d(y, u)| \leq \theta^{\frac{7}{2}}$. Finally, Lemma 3.28 implies that $d(z, x) \leq \theta^{\frac{7}{2}}$. Thus, the distance between w and $\check{\Sigma}_x$ is at most $2\theta^{\frac{7}{2}}$. (Here, we use again that close to x the cross section $\Sigma_{x;a,b}$ is almost 1-dimensional, i.e. close to an interval.)

All in all, we conclude that $d(u, \check{\Sigma}_x) \leq \theta^{\frac{99}{30}}$. By switching the roles of $\check{\Sigma}_x$ and $\check{\Sigma}_y$, we similarly obtain that every point in $\check{\Sigma}_x$ has distance $\leq \theta^{\frac{99}{30}}$ to $\check{\Sigma}_y$. This completes the proof. $\quad\square$For future reference, we observe that the lemma also holds true if we replace $\check{\Sigma}_y$ by $B(y, \frac{1}{2}\tau\theta^3)$ for some $\tau \in (1 - \theta, 1 + \theta)$. The proof for (i) goes through unchanged and for (ii) it suffices to observe that the almost 1-dimensionality of $\Sigma_{y;a',b'}$ near y (Lemma 3.21) implies $d_H(\check{\Sigma}_y, B(y, \frac{1}{2}\tau\theta^3)) < \theta^{\frac{99}{30}}$.

3.6 Necks

A neck occurs where the connected component of a cross section has small diameter.

Definition 3.30 (Necklike points). A point $y \in B(x, 5)$ is called θ-*necklike* relative to a $< 2\theta$-straight 1-strainer (a, b) of length $> (1 - \theta)$ (at y), if $\operatorname{diam}(\Sigma_{y;a,b}) < \theta^2$.

We say that $y \in B(x, 5)$ is θ-*weakly necklike* relative to a $< 2\theta$-straight 1-strainer (a, b) of length $> (1 - \theta)$ at y if $\mathrm{diam}(\Sigma_{y;a,b}) < 2\theta^2$ and that it is θ-*strongly necklike* relative to such a strainer if $\mathrm{diam}(\Sigma_{y;a,b}) < \frac{3}{4}\theta^2$.

Our new definition allows us to reformulate part (ii) of Lemma 3.28: Suppose that for an edgy point x, every point in $B(x, \theta)$ admits a $< 2\theta$-straight 1-strainer of length $> (1 - \theta)$ Then every point $y \in axb \cap B(x, \theta^3)$ is θ-strongly necklike or has distance $< \theta^{\frac{15}{4}}$ from a point z which is θ-strongly edgy.

Suppose that x is θ-necklike relative to (a, b). Nearby cross sections have comparable diameters: Using the metric properties of the maps $\Phi_t^{a,b}$ one sees that

$$\mathrm{diam}(\Sigma_{z;a,b}) < (1 + c\theta)\theta^2 + c\theta |f_{a,b}(z) - f_{a,b}(y)| < c'\theta^2$$

for $z \in B(y, \theta)$.

Every segment of length $> \theta$ initiating in $B(y, \theta)$ must pass through one of the two cross sections $f_{a,b}^{-1}(f_{a,b}(y) \pm \frac{9}{10}\theta)$. Hence, triangle comparison and (3.9) imply for any point a' with $d(y, a') > \theta$ that

$$\angle_z(a, a'), \angle_z(b, a') \notin [c\theta, \pi - c\theta] \tag{3.31}$$

for $z \in B(y, \frac{\theta}{2})$. In particular, any $\frac{\pi}{2}$-straight 1-strainer (a', b') of length $> \theta$ at a point in $B(y, \frac{\theta}{2})$ is $c'\theta$-straight, and $f_{a',b'} - f_{a,b}$ is $c''\theta$-Lipschitz on $B(y, \frac{\theta}{2})$.

If x is θ-necklike relative to a 1-strainer (a, b) with $\mathrm{diam}\,\Sigma_{x;a,b} < \frac{1}{2}\theta^2$, and if all points in $B(x, \theta)$ admit $< 2\theta$-straight 1-strainers of length $> (1 - \theta)$ we can conclude from the above observations as in the proof of Lemma 3.26 that all points in $B(x, \theta)$ are also θ-strongly necklike.

We now deduce that cross sections of nearby necklike points are almost parallel.

Lemma 3.32 (Almost parallel cross sections of necks). *Let x be θ-necklike relative to (a, b). Furthermore, let $y \in B(x, \theta)$ be θ-weakly necklike with respect to a 1-strainer (a', b').*

(i) If $\Sigma_{x;a,b}$ and $\Sigma_{y;a',b'}$ have nonempty intersection, then $d(x, y) < c'\theta^3$

(ii) If $d(x, y) < \theta^{\frac{11}{6}}$, then the Hausdorff distance of $\Sigma_{x;a,b}$ and $\Sigma_{y;a',b'}$ is less that $\theta^{\frac{5}{3}}$.

Proof. The proof is closely related to the one for edges, i.e. 3.29.

(i) Let z be one of the intersection points of the cross sections $\Sigma_{x;a,b}$ and $\Sigma_{y;a',b'}$. By our discussion above, $f_{a,b} - f_{a',b'}$ is $c\theta$-Lipschitz on $B(z, \theta^2)$, and hence $f_{a,b}$ is $c\theta^{\frac{1}{2}}$-Lipschitz on $\Sigma_{y;a',b'}$. It follows that $|f_{a,b}(x) - f_{a,b}(y)| = |f_{a,b}(z) - f_{a,b}(y)| < c'\theta^3$.

(ii) Consider a point $u \in \check{\Sigma}_y$. By 3.31, we have $\angle_y(a, a') \leq c'\theta$. When projecting $\Sigma_{y;a',b'}$ to $\Sigma_{y;a,b}$, we map u to a point v with $d(u, v) \leq c''\theta^3$ by 3.14. The coarse flow $\Phi_{-f_{a,b}(y)}^{a,b}$ transports $v \in \Sigma_{y;a,b}$ to some point $w \in \Sigma_{x;a,b}$ with $d(v, w) \leq c'''\theta^{\frac{11}{6}}$.

This shows that $d(u, \Sigma_{x;a,b}) \leq \theta^{\frac{5}{3}}$. Again, we switch the roles of $\Sigma_{x;a,b}$ and $\Sigma_{y;a',b'}$ to complete the proof. $\square$

4 Locally volume collapsed 3-orbifolds are graph

4.1 Setup and formulation of main result

Let (O, g) be a closed connected Riemannian 3-orbifold which does *not* have nonnegative sectional curvature, $sec \not\geq 0$.

Definition 4.1 (Curvature scale). For $-b^2 \in [-1, 0)$ we define the $-b^2$-*(sectional) curvature scale* in a point $x \in O$ as the maximal radius $\rho_{-b^2}(x) \in (0, \infty)$ such that the rescaled ball $B_{\rho_{-b^2}(x)^{-2}g}(x, 1) = \rho_{-b^2}(x)^{-1} \cdot B_g(x, \rho_{-b^2}(x))$ has sectional curvature $sec \geq -b^2$.

Note that

$$b\rho_{-1} \leq \rho_{-b^2} \leq \rho_{-1}. \tag{4.2}$$

The function ρ_{-b^2} is continuous on O. More precisely, ρ_{-b^2} *does not oscillate too fast* in the sense that for $0 < \lambda < 1$ holds

$$(1 - \lambda)\rho_{-b^2}(x) \leq \rho_{-b^2} \leq (1 + \lambda)\rho_{-b^2}(x) \tag{4.3}$$

on $B(x, \lambda\rho_{-b^2}(x))$.

The rescaled balls $B_{\rho_{-b^2}(x)^{-2}g}(x, 1)$ are Alexandrov balls with curvature $\geq -b^2$ and radius ≤ 1 in the sense of definition 3.1.

The purpose of this paper is to study the geometry and topology of 3-orbifolds which are locally collapsed relative to the curvature scale.

Definition 4.4 (Local volume collapse). Let $v > 0$ and let $\sigma : O \to (0, \infty)$ be some (not necessarily continuous) function. We say that (O, g) is v-*collapsed* at the scale σ, if for all points x holds $vol(B_{\sigma(x)^{-2}g}(x, 1)) < v$, equivalently, $vol(B_g(x, \sigma(x))) < v\sigma(x)^3$.

If $sec \not\geq 0$, we say that (O, g) is $(v, -b^2)$-*collapsed*, if it is v-collapsed at the scale ρ_{-b^2}.

Note that if (O, g) is locally v-collapsed at some scale $\sigma \leq \rho_{-b^2}$, then Bishop-Gromov volume comparison yields that it is locally $(v', -b^2)$-collapsed with $v' = \frac{vol(B_{-b^2}(1))}{vol(B_0(1))}v \leq \frac{vol(B_{-1}(1))}{vol(B_0(1))}v$ (independent of $-b^2$!). Here $B_{-b^2}(1)$ denotes the unit 3-ball with $sec \equiv -b^2$.

Strongly volume collapsed Riemannian 3-orbifolds are, on the scale of their collapse, close to Alexandrov spaces of dimension ≤ 2 and with curvature $\geq -b^2$, and the volume collapse translates into the shortness of $2\frac{1}{2}$-strainers, cf. section 3.1.2).

Lemma 4.5. *For $\lambda > 0$ exists $v = v(\lambda) > 0$ such that:*

If (O, g) is $(v, -b^2)$-collapsed and $x \in O$, then $\bar{\theta}_{2\frac{1}{2}}$-straight $2\frac{1}{2}$-strainers in the Alexandrov ball $\rho_{-b^2}(x)^{-1}B(x, \rho_{-b^2}(x))$ of curvature $\geq -b^2 \geq -1$ have length $< \lambda$.

Proof. Suppose that the Riemannian 3-orbifolds (O_i, g_i) are $(\frac{1}{i}, -b_i^2)$-collapsed but contain points x_i which admit such that there are $\bar{\theta}_{2\frac{1}{2}}$-straight $2\frac{1}{2}$-strainers of length $\geq \lambda$ in the balls $\rho_{-b_i^2}(x)^{-1}B(x, \frac{1}{2}\rho_{-b_i^2}(x))$. Then the rescaled balls $\rho_{-b^2}(x_i)^{-1}B(x_i, 1)$ Gromov-Hausdorff subconverge to an Alexandrov ball with dimension ≤ 2 and curvature ≥ -1 which admits $\bar{\theta}_{2\frac{1}{2}}$-straight $2\frac{1}{2}$-strainers of length λ, a contradiction. $\square$

We will require some additional regularity for our Riemannian orbifolds. The conclusions on the global topology of collapsed 3-orbifolds are valid without this regularity condition, but it is technically convenient because it avoids the use of (an orbifold version of) Perelman's Stability Theorem for Alexandrov spaces and is expected to be satisfied by the output of the Ricci flow on 3-orbifolds. In the next definition, ω_3 denotes the volume of the euclidean unit 3-ball. (Compare [Pe03, 7.4] and [KL10, Thm. 1.3].)

Definition 4.6 (Local curvature control). Fix numbers $s_0 \in \mathbb{N}$, $v_0 \in (0, \omega_3)$, a function $K : (0, \omega_3) \to (0, \infty)$ and a scale function $\sigma : O \to (0, \infty)$. We say that (O, g) has (v_0, s_0, K)-*curvature control* below scale σ, if the following holds: If $\mathrm{vol}\, B(x, r) \geq vr^3$ for $v \in [v_0, \omega_3)$ and $r \in (0, \sigma(x)]$, then $\|\nabla^s R\| \leq K(v) r^{-2-s}$ on $B(x, r)$, equivalently, $\|\nabla^s R\| \leq K(v)$ on the rescaled ball $r^{-1} \cdot B(x, r)$ for $s = 0, \dots, s_0$.

We will apply this notion in the following situation.

Lemma 4.7. *Let (O_i, g_i) be a sequence of Riemannian 3-orbifolds as above with (v_i, s_0, K)-curvature control below scale ρ_{-b^2}, where $v_i \to 0$. Furthermore, let $x_i \in O_i$ be points and $\lambda_i \to 0$ positive numbers.*

Then, if s_0 is sufficiently large, the sequence of rescaled pointed orbifolds $(\lambda_i \rho_{-b^2}(x_i))^{-1} \cdot (O_i, x_i)$ subconverges either in the Gromov-Hausdorff sense to an Alexandrov space with curvature ≥ 0 and dimension ≤ 2, or in the $\mathcal{C}^5$-topology to a $\mathcal{C}^{10}$-smooth complete Riemannian 3-orbifold with $\sec \geq 0$.

Proof. Clearly, we have Gromov-Hausdorff subconverge to a pointed Alexandrov space (X, x_0) with curvature ≥ 0 and dimension ≤ 3. If $\dim(X) = 3$, the convergence can be improved using our assumption of local curvature control. The approximating pointed 3-orbifolds $(\lambda_i \rho_{-b^2}(x_i))^{-1} \cdot (O_i, x_i)$ are uniformly noncollapsed. Indeed, for any $r > 0$, we have $\mathrm{vol}\, B_{\lambda_i^{-2} \rho_{-b^2}(x_i)^{-2} g_i}(x_i, r) > \frac{1}{2} \mathrm{vol}_3 B(x_0, r) > 0$ for large i, where volume in X is measured with respect to the 3-dimensional Hausdorff measure. Thus the (v_i, s_0, K)-curvature control on $(\lambda_i \rho_{-b^2}(x_i))^{-1} \cdot O_i$ below the scales $\lambda_i^{-1} \to \infty$ applies for large i and yields on the balls $B_{\lambda_i^{-2} \rho_{-b^2}(x_i)^{-2} g_i}(x_i, r)$ uniform bounds on the curvature tensor and its covariant derivatives up to order s_0. The smoothness of the limit and the convergence follow from an orbifold version of well-known compactness results for pointed Riemannian manifolds with bounds on curvature and some of its derivatives, e.g. from the following special case of a result in [FL]:

Theorem 4.8. *Let $r_0, v_0 > 0$ and let $C : (0, \infty) \to (0, \infty)$ be a function. Then the space of pointed complete $\mathcal{C}^\infty$-smooth Riemannian 3-orbifolds (O, p) such that $\mathrm{vol}\, B(p, r_0) \geq v_0$ and such that on every ball $B(p, r)$ around p the curvature tensor and its covariant derivatives up to order 20 on $B(p, r)$ are bounded by the constant $C(r)$, is precompact in the pointed $\mathcal{C}^{15}$-topology. Thus, every sequence of such orbifolds (O_i, p_i) subconverges $\mathcal{C}^5$-smoothly to a $\mathcal{C}^{10}$-smooth Riemannian 3-orbifold.*

This completes the proof of 4.7. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The main result of this paper is (compare [Pe03, Theorem 7.4], [MT08, Theorem 0.2] and [KL10, Theorem 1.3]):

Theorem 4.9. *Let $s_0 \in \mathbb{N}$ and let $K : (0, \omega_3) \to (0, \infty)$ be a function. If s_0 is sufficiently large, then there exists a constant $v_0 = v_0(s_0, K) \in (0, \omega_3)$ such that:*

If (O, g) is $(v_0, -1)$-collapsed, has (v_0, s_0, K)-curvature control below the scale ρ_{-1} and contains no bad 2-suborbifolds, then O admits a metric with $\sec \geq 0$ or can be decomposed by finitely many surgeries into components which are spherical or graph.

Remark 4.10. Unlike in the manifold case we cannot conclude that O is always graph, because there are non-graph 3-orbifolds admitting nonnegatively curved (e.g. spherical or euclidean) metrics.

One can reduce to the case of a lower diameter bound relative to a curvature scale by suitably rescaling. 4.9 follows from:

Theorem 4.11. *Let $s_0 \in \mathbb{N}$ and let $K : (0, \omega_3) \to (0, \infty)$ be a function. If s_0 is sufficiently large, then there exists a constant $v_0 = v_0(s_0, K) \in (0, \omega_3)$ such that:*

If for some $-b^2 \in [-1, 0)$ the orbifold (O, g) is $(v_0, -b^2)$-collapsed, satisfies $\mathrm{rad}(O, \cdot) \geq \frac{1}{2}\rho_{-b^2}$, has (v_0, s_0, K)-curvature control below the scale ρ_{-b^2}, and contains no bad 2-suborbifolds, then O can be decomposed by finitely many surgeries into components which are spherical or graph.

Proof that 4.11 implies 4.9. Suppose that (O_i) is a sequence of $(v_i, -1)$-collapsed orbifolds with (v_i, s_0, K)-curvature control below the scale ρ_{-1} where $v_i \to 0$. Then we must show that the O_i satisfy the conclusion of 4.9 for infinitely many i.

Note that for all $b \in (0, 1]$ the O_i are $(b^{-3}v_i, -b^2)$-collapsed and have (v_i, s_0, K)-curvature control below the scale ρ_{-b^2}, cf. (4.2). Hence we are done, if for some $-b^2 \in [-1, 0)$ holds $\mathrm{rad}(O_i) \geq \frac{1}{2}\rho_{-b^2}$ for all sufficiently large i.

Otherwise, after passing to a subsequence, there exist sequences of numbers $-b_i^2 \to 0$ and points $x_i \in O_i$ such that $\mathrm{rad}(O_i, x_i) < \frac{1}{2}\rho_{-b_i^2}(x_i)$. It follows that $\rho_{-b_i^2} \equiv const_i \geq \mathrm{diam}(O_i)$ and we have collapse to the point in the sense of $\mathrm{diam}(O_i) \cdot (-\min \sec_{O_i})^{\frac{1}{2}} \to 0$. We rescale and increase the $-b_i^2$ so that $\mathrm{diam}(O_i) = 1$ and $\min \sec_{O_i} = -b_i^2 \to 0$. Then $\rho_{-b_i^2} \equiv 1$.

If $v_i' = \mathrm{vol}(O_i) \to 0$, then the O_i are $(v_i', -b_i^2)$-collapsed with (v_i, s_0, K)-curvature control below the scales $\rho_{-b_i^2}$ and with $\mathrm{rad}(O_i) \geq \frac{1}{2} \equiv \frac{1}{2}\rho_{-b_i^2}$, and we are done by 4.11.

Otherwise, after passing to a subsequence, we have a lower volume bound $\mathrm{vol}(O_i) \geq v' > 0$ and, due to the curvature control, uniform (global) bounds on the curvature tensor and its covariant derivatives up to order s_0. If s_0 is large enough, it follows that the O_i subconverge, say, in the $\mathcal{C}^5$-topology to a $\mathcal{C}^5$-Riemannian 3-orbifold O_∞ with $\sec \geq 0$. In particular, infinitely many O_i are diffeomorphic to O_∞ and therefore admit metrics with $\sec \geq 0$. $\square$

We fix some arbitrary (more than) sufficiently large value for s_0, say $s_0 := 2010$.

For the rest of this section we make the following assumption on the orbifolds we work with:

Assumption 4.12. *(O, g) is a closed connected Riemannian 3-orbifold such that $\sec \not\geq 0$ and $\mathrm{rad}(O, \cdot) \geq \frac{1}{2}\rho_{-b^2}$ where $-b^2 \in [-1, 0)$.*

Together with Corollary 2.9, Theorem 4.9 implies the following

Theorem 4.13. *Let $s_0 \in \mathbb{N}$ and let $K : (0, \omega_3) \to (0, \infty)$ be a function. If s_0 is sufficiently large, then there exists a constant $v_0 = v_0(s_0, K) \in (0, \omega_3)$ such that:*

If (O, g) is closed and $(v_0, -1)$-collapsed, has (v_0, s_0, K)-curvature control below the scale ρ_{-1} and contains no bad 2-suborbifolds, then O either admits a metric with $\sec \geq 0$, or satisfies Thurston's Geometrization Conjecture.

4.2 Conical approximation and humps

Sufficient local volume collapse leads to good local approximation by cones of dimension ≤ 2 on scales comparable to the curvature scale. Proposition 3.4 for $d = 2$ and Lemma 4.5 imply:

Proposition 4.14 (Uniform conical approximation relative to the curvature scale). *For $0 < \sigma, \mu < 1$ exist a scale $0 < s_1 = s_1(\sigma, \mu) << \sigma$ and a rate of collapsedness $v = v(\sigma, \mu) > 0$ such that the following holds:*

If (O, g) is $(v, -b^2)$-collapsed, then it can in every point x be μ-well approximated (cf. 3.3) on some scale $s(x) \in [s_1 \rho_{-b^2}(x), \sigma \rho_{-b^2}(x)]$ by a cone of dimension 1 or 2, i.e. by the open interval $(-1, 1)$, by the half-open interval $[0, 1)$, or by the cone of radius 1 over a circle or an interval of diameter $\leq \pi$.

Proof. The assertion follows with $s_1 = s_1(\sigma, \mu) := s_1(2, \sigma, \mu)$ from Proposition 3.4 and $v = v(\sigma, \mu) := v(s_{2\frac{1}{2}}(\sigma, \mu))$ from Lemma 4.5. $\qquad\square$

We fix some small value $\sigma \in (0, \frac{1}{10})$ for the upper bound on the local scales of approximation.

Next we wish to divide our orbifold into regions with and without good 1-strainers. The region with good 1-strainers has locally almost product geometry. It consists of the points where the bases of approximating cones provided by 4.14 have diameter $\approx \pi$.

For small $\theta > 0$, let $\mu_0(\theta) > 0$ be the constant given by 3.6. For $\mu \in (0, \mu_0(\theta)]$, we call a point $x \in O$ a $(\theta, \mu, -b^2)$-*hump*, if O can in x be μ-well approximated on the scale $s(x)$ by a cone with base of diameter $< \pi - \frac{\theta}{2}$, i.e. by the half-open interval $[0, 1)$ or the cone of radius 1 over a circle or an interval with diameter $< \pi - \frac{\theta}{2}$. In other words, x is a (θ, μ)-hump of the rescaled ball $B_{\rho_{-b^2}(x)^{-2}g}(x, 1)$ in the sense of Definition 3.5 for $d = 2$. If x is no $(\theta, \mu, -b^2)$-hump, then O can in x be μ-well approximated on the scale $s(x)$ by a cone with base of diameter $\geq \pi - \frac{\theta}{2}$, i.e. by the open interval $(-1, 1)$ or the cone of radius 1 over a circle or an interval with diameter $\geq \pi - \frac{\theta}{2}$.

We denote by $H = H_{\theta, \mu, -b^2} \subset O$ the subset of $(\theta, \mu, \rho_{-b^2}(x))$-humps and by $S = S_{\theta, \mu, -b^2} \subset O$ the open subset of points x admitting equilateral 1-strainers which are $< \theta$-straight with length in $(\frac{1}{11}s_1(\sigma, \mu), \frac{3}{22}s_1(\sigma, \mu))$ in the Alexandrov ball $(\rho_{-b^2}(x))^{-1}B(x, \rho_{-b^2}(x))$ of curvature ≥ -1.

Throughout the following chapters, we will abbreviate this property by saying that the points $x \in S_{\theta, \mu, -b^2}$ admit $< \theta$-straight (equilateral) 1-strainers with length in the interval $(\frac{1}{11}s_1(\sigma, \mu)\rho_{-b^2}(x), \frac{3}{22}s_1(\sigma, \mu)\rho_{-b^2}(x))$.

Due to our bound on the approximation accuracy μ, the implications of 3.6 hold: A $(\theta, \mu, -b^2)$-hump x admits no $\frac{\theta}{4}$-straight 1-strainers of length $\frac{1}{111}s(x)$, but all points in the closed annulus $\overline{A}(x; \frac{1}{10}s(x), \frac{9}{10}s(x))$ do admit $\frac{\theta}{11}$-straight 1-strainers of length $> \frac{1}{11}s(x) \geq$

$\frac{1}{11}s_1(\sigma,\mu)\rho_{-b^2}(x)$, i.e. $\overline{A}(x;\frac{1}{10}s(x),\frac{9}{10}s(x)) \subset S_{\theta,\mu,-b^2}$. On the other hand, if x is no $(\theta,\mu,-b^2)$-hump, then it admits $< \theta$-straight 1-strainers of length $> \frac{99}{100}s(x) > \frac{1}{11}s_1(\sigma,\mu)\rho_{-b^2}(x)$. Hence $O = H_{\theta,\mu,-b^2} \cup S_{\theta,\mu,-b^2}$.

Proposition 4.15 (cf. 3.7). *If $\mu \leq \mu_0(\theta)$ and (O,g) is $(v(\sigma,\mu),-b^2)$-collapsed (cf. 4.14), then there exist finitely many $(\theta,\mu,-b^2)$-humps $x_j \in H_{\theta,\mu,-b^2}$ such that*

$$O = \left(\bigcup_j B(x_j,\frac{1}{10}s(x_j))\right) \cup S_{\theta,\mu,-b^2}.$$

Moreover, $d(x_j,x_k) > \frac{9}{10}s(x_j)$ for $j \neq k$.

Proof. We proceed as in the proof of 3.7.

Again, there can be no infinite sequence of points $x_1, x_2, \cdots \in H_{\theta,\mu,-b^2} - S_{\theta,\mu,-b^2}$ such that for all k holds $x_k \in B(x_{k+1},\frac{1}{10}s(x_{k+1}))$ and $x_{k+1} \notin B(x_k,\frac{1}{10}s(x_k))$, because then $s(x_1) < \frac{1}{9^{k-1}}s(x_k)$. But due to the continuity of the curvature scale ρ_{-b^2}, cf. (4.3), and the compactness of O the scales take also in this situation values in a bounded interval, namely in $[s_1 \min \rho_{-b^2}, \sigma \min \rho_{-b^2}]$.

The rest of the proof goes through without change. $\qquad\square$

4.3 The Shioya-Yamaguchi blow-up

We recall the Shioya-Yamaguchi blow-up argument, see [SY00, §3, Key Lemma 3.6]. To simplify things, we restrict ourselves to certain special situations. Some of our arguments are different; we also treat some additional cases not mentioned there explicitely.

4.3.1 General discussion

Consider the following situation. Let $B(p_i,1)$ be a sequence of d-dimensional Riemannian orbifold balls (i.e. open metric balls of radius 1 in complete Riemannian d-orbifolds without boundary) with curvature $sec \geq -1$ which collapse to an Alexandrov ball of strictly smaller dimension $1 \leq k < d$,

$$B(p_i,1) \longrightarrow X = B(x,1). \tag{4.16}$$

We suppose furthermore that the collapse limit X is $(0-)$*conelike* in the sense that every segment initiating in x extends to length 1, and that the closed balls $\overline{B}(p_i,\frac{1}{2})$ are *not discal*. Note that the conelikeness of X implies that for any fixed $\epsilon > 0$ the annulus $A(p_i,\epsilon,1-\epsilon)$ contains for large i no critical points of the distance function $d(p_i,\cdot)$ and in particular $\overline{B}(p_i,\frac{1}{2})$ is a topological suborbifold with boundary.

Let $\hat{p}_i \in B(p_i,1)$ be any sequence of points with $d(\hat{p}_i,p_i) \to 0$. The distance function $d(\hat{p}_i,\cdot)$ must have critical values in $(0,\frac{1}{2})$, because $\overline{B}(p_i,\frac{1}{2})$ is not discal. Let δ_i be the maximal critical value in $(0,\frac{1}{2})$, and let $q_i \in B(p_i,\frac{1}{2})$ be a critical point at distance $d(\hat{p}_i,q_i) = \delta_i$ from $\hat{p}_i$. Then $\delta_i \to 0$ because X is conelike. For any constant $c > 1$ holds that

$$\overline{B}(p_i,\frac{1}{2}) \cong \overline{B}(\hat{p}_i,c\delta_i) \tag{4.17}$$

for sufficiently large i, i.e. the topology of the balls $B(p_i, 1)$ is concentrated near their centers. (Note that, also due to the conelikeness of X, there exists a common gradient like vector field for $d(p_i, \cdot)$ and $d(\hat{p}_i, \cdot)$, and so $\overline{B}(p_i, \frac{1}{2}) \cong \overline{B}(\hat{p}_i, \frac{1}{2})$.)

To help revealing the local topology at the p_i, we form (modulo passing to a subsequence) the *blow-up limit*

$$(\delta_i^{-1} B(p_i, 1), \hat{p}_i) \longrightarrow (Y, y_0). \tag{4.18}$$

The limit space Y is a noncompact Alexandrov space with dimension $\geq k$ and curvature ≥ 0. In particular, the Soul Theorem applies. Moreover $q_i \to z$ with z a critical point of $d(y_0, \cdot)$ at distance $d(y_0, z) = 1$.

If no collapse happens any more in the blow-up limit (4.18), i.e. if $\dim Y = d$, then we need topological stability in order to relate the topologies of the balls $B(p_i, 1)$ and Y. For instance, if Y and the convergence in (4.18) are sufficiently smooth as will be the case in the situations considered later in the paper, say Y is $\mathcal{C}^{10}$-smooth and the convergence is $\mathcal{C}^5$-smooth, then one can argue as follows. There exist $r, \epsilon > 0$ and a smooth vector field V on $Y - \overline{B}(y_0, \frac{r}{2})$ such that for all $y \notin \overline{B}(y_0, \frac{r}{2})$ the vector $V(y)$ has angles $\geq \frac{\pi}{2} + \epsilon$ with all segments yy_0, compare the proof of the Soul Theorem. We regard $\delta_i^{-1} B(\hat{p}_i, 2r\delta_i)$ as embedded in Y for large i. In particular, $\hat{p}_i \to y_0$ and V is on a neighborhood of $\partial B(y_0, r)$ gradient-like not only for $d(y_0, \cdot)$ but also for $\delta_i^{-1} d_{B(p_i,1)}(\hat{p}_i, \cdot)$, viewed as a function on part of Y. Hence $\delta_i^{-1} \overline{B}(\hat{p}_i, r\delta_i)$ is isotopic to $\overline{B}(y_0, r)$ in Y, and with (4.17) we see that $\overline{B}(p_i, \frac{1}{2})$ is for large i homeomorphic to the closed disc bundle in the normal bundle of the soul of Y, in other words, to a (small) closed tubular neighborhood of the soul.

The blow-up limit (4.18) is in general still a collapse to lower dimension. The aim of the following discussion is to find situations when the drop of dimension is strictly smaller than for the original collapse (4.16).

The following construction is as in the proof of Soul Theorem: Let $\xi \in \Sigma_x X$ be a direction at x. Due to conelikeness it is represented by a (unique) segment σ_ξ emanating from x. Fix some $t_0 \in (0, 1)$, say $t_0 = \frac{1}{10}$, and let $\sigma_\xi^i \in B(p_i, 1)$ be a sequence of points converging to the point $\sigma_\xi(t_0)$ on σ_ξ at distance t_0 from x. (Our choice of the $\sigma_\xi(t_0)$ is independent of the choice of the $\hat{p}_i$.) The segments $\hat{p}_i \sigma_\xi^i$ subconverge to a ray ρ_ξ in Y emanating from y_0. Moreover, the normalized distance functions $\delta_i^{-1}(d(\sigma_\xi^i, \cdot) - d(\sigma_\xi^i, \hat{p}_i))$ on the rescaled balls $\delta_i^{-1} B(p_i, 1)$ subconverge (due to Arzela-Ascoli) to a concave 1-Lipschitz function β_ξ on Y with $\beta_\xi(y_0) = 0$ which decays along ρ_ξ with extremal slope -1, $\beta_\xi(\rho_\xi(t)) = -t$, where we use a unit speed parametrization $\rho_\xi(t)$ starting at $\rho_\xi(0) = y_0$. In fact, every point $y \in Y$ is the initial point of a ray ρ_ξ^y along which β_ξ decays with slope -1. In particular, the level sets of β_ξ have no interior points. The comparison of β_ξ with the Busemann function $b_\xi = \lim_{t \to \infty} (d(\rho_\xi(t), \cdot) - d(\rho_\xi(t), y_0))$ associated to the ray ρ_ξ is given by the inequality

$$\beta_\xi \leq b_\xi.$$

To verify this, let $\sigma_\xi^i(a)$ denote the point on $\hat{p}_i \sigma_\xi^i$ at (unrescaled) distance $a\delta_i$ from $\hat{p}_i$. Then $d(\sigma_\xi^i, \cdot) - d(\sigma_\xi^i, \hat{p}_i) \leq d(\sigma_\xi^i(a), \cdot) - d(\sigma_\xi^i(a), \hat{p}_i)$ for large i and hence $\beta_\xi \leq d(\rho_\xi(a), \cdot) - a$ for all $a > 0$. Letting $a \to \infty$ yields the inequality. As a consequence, the convex suplevel sets of β_ξ are smaller (not larger) than the corresponding suplevels of b_ξ.

Since the q_i are critical for $d(\hat{p}_i, \cdot)$, we have that $\tilde{\angle}_{q_i}(\hat{p}_i, \sigma_\xi^i) \leq \frac{\pi}{2}$ and $\liminf_{i \to \infty} \delta_i^{-1}(d(\sigma_\xi^i, q_i) -$

$d(\sigma_\xi^i, \hat{p}_i)) \geq 0$. So

$$\beta_\xi(z) \geq 0$$

and z is contained in the totally convex subset $\cap_\xi \{\beta_\xi \geq 0\} = \{\min_\xi \beta_\xi \geq 0\}$.

The blow-up *expands* $\Sigma_x X$ in the following sense. When doing the above construction for two directions ξ and ξ' at the same time, one obtains for every point $y \in Y$ a pair of rays ρ_ξ^y and $\rho_{\xi'}^y$ satisfying

$$\angle_{Tits}(\rho_\xi^y, \rho_{\xi'}^y) := \lim_{t \to \infty} \tilde{\angle}_y(\rho_\xi^y(t), \rho_{\xi'}^y(t)) \geq \angle_x(\xi, \xi'). \tag{4.19}$$

The construction can be done for any finite subset $A \subset \Sigma_x X$ and hence yields weakly expanding maps $\epsilon_{y,A} : A \to \partial_{Tits} Y$.

We will use the following observations:

Lemma 4.20. *(i) The blow-up limit Y is not isometric to a euclidean space.*

(ii) If $\Sigma_x X$ contains an embedded unit l-sphere (i.e. with $\sec \equiv 1$), then so does $\partial_{Tits} Y$ and Y splits off an $\mathbb{R}^{l+1}$-factor. If $\Sigma_x X$ contains an embedded unit l-hemisphere, then so does $\partial_{Tits} Y$ and Y contains an isometrically embedded copy of the $(l+1)$-dimensional euclidean halfspace (and in particular splits off an $\mathbb{R}^l$-factor).

Proof. (i) $d(y_0, \cdot)$ has a critical point (at distance 1).

(ii) Choose A as the union of $l + 1$ pairs of antipodes which span the embedded unit sphere (corresponding to coordinate axes). Then the expanding map $A \to \partial_{Tits} Y$ must be an isometric embedding and the assertion follows from the Splitting Theorem. The second assertion follows similarly by applying the first assertion to the boundary $(l - 1)$-sphere of the embedded l-hemisphere. $\square$

4.3.2 The case of flat conical limits with dimension ≤ 2

We apply the general discussion above in certain special situations. Note that always $\dim Y \geq \dim X$. We aim now to achieve that $\dim Y > \dim X$ by making a good choice of the $\hat{p}_i$.

Collapse to a flat k-disc. Suppose that X is isometric to the euclidean unit k-disc, $k \geq 1$. By 4.20, Y splits off an $\mathbb{R}^k$-factor and $Y \not\cong \mathbb{R}^k$. Hence always $\dim(Y) > k$, independently of the choice of the $\hat{p}_i$.

Noses: Collapse to the half-open interval. Suppose that $X = [0, 1)$ with $x = 0$. There is a unique direction ξ at $x = 0$. We choose $\hat{p}_i$ as a "tip" of the nose, i.e. as a maximum of $d(\sigma_\xi^i, \cdot)$. Then $\hat{p}_i \to x$ and the choice of the $\hat{p}_i$ is admissible in the sense that $d(p_i, \hat{p}_i) \to 0$. The base point y_0 is a maximum of β_ξ and hence $\beta_\xi(z) = \beta_\xi(y_0) = 0$. If $\dim(Y) = 1$, then Y is a halfline since $Y \not\cong \mathbb{R}$ by 4.20(i), and $\beta_\xi = b_\xi$ has a unique maximum. This contradicts $z \neq y_0$. Thus $\dim(Y) \geq 2$.

Note that Y contains a flat half-strip, but nevertheless its geometry is in general not rigid.

Collapse to a flat 2-disc with cone point or a sector. Suppose that X is the cone of radius 1 over a circle or interval with diameter $< \pi$. We generalize the cases of noses to humps by adapting the argument in [SY00, §3] to this case.

Let $A \subset \Sigma_x X$ be a finite subset such that $\sum_{\xi \in A} d(\sigma_\xi(t_0), \cdot)$ has a unique maximum in x. We choose the $\hat{p}_i$ as maxima of the corresponding functions $\sum_{\xi \in A} d(\sigma_\xi^i, \cdot)$. Then $\hat{p}_i \to x$. The point y_0 is a maximum of $\sum_{\xi \in A} \beta_\xi$. It follows that $\beta_\xi(z) = 0$ for all $\xi \in A$, and the totally convex subset $\cap_{\xi \in A} \{\beta_\xi \geq 0\} = \cap_{\xi \in A} \{\beta_\xi = 0\}$ containing y_0 and z has positive dimension. In particular, $\dim Y \geq 2$.

Suppose that $\dim Y = 2$. Then $\cap_{\xi \in A} \{\beta_\xi = 0\}$ is one-dimensional. Let y be an interior point of a segment $y_0 z$. Since the rays ρ_ξ^y for $\xi \in A$ are perpendicular to $y_0 z$, there can be at most two of them, $|A| \leq 2$. Since we are free to choose $|A|$ with any cardinality, we obtain a contradiction. Thus $\dim Y \geq 3$.

An argument analogous to the last one shows furthermore that the soul of Y must have codimension ≥ 2. In particular, if $\dim Y = 3$, then $\dim soul(Y) \leq 1$.

Collapse to the flat 2-halfdisc. Suppose that X is the flat unit halfdisc in $\{u \in \mathbb{R}^2 : u_2 \leq 0\}$ centered at $x = 0$. (This case has not been treated explicitly in [SY00, §3]. There, blow-up limits have been obtained under the assumption that $\operatorname{diam}(\Sigma_x X) < \pi$.)

4.20 implies that Y contains a flat halfplane, but $Y \ncong \mathbb{R}^2$. In particular, Y splits metrically as $Y \cong \mathbb{R} \times W$. If $\dim Y = 2$, then W is a halfline and Y a flat halfplane. If $\dim Y = 3$, then W is a noncompact Alexandrov surface with curvature ≥ 0. We may assume that $y_0 \in 0 \times W$. The critical points of $d(y_0, \cdot)$ lie on $0 \times W$.

If we denote by $\eta^\pm \in \Sigma_x X$ the directions pointing to $(\pm 1, 0)$, then for any $y \in Y$ the rays $\rho_{\eta^\pm}^y$ have angle π at y, cf. (4.19), and their union is the line $\mathbb{R} \times w$ through y. Moreover, $\{\beta_{\eta^+} = \beta_{\eta^-}\} = 0 \times W$, and it is the Gromov-Hausdorff limit of the bisectors $\{d(\sigma_{\eta^+}^i, \cdot) = d(\sigma_{\eta^-}^i, \cdot)\}$.

For any direction $\xi \in \Sigma_x X$ holds $\angle_y(\rho_\xi^y, \rho_{\eta^\pm}^y) = \angle_x(\xi, \eta^\pm)$ because $\pi = \angle_{Tits}(\rho_{\eta^+}^y, \rho_\xi^y) + \angle_{Tits}(\rho_\xi^y, \rho_{\eta^-}^y) \geq \angle_y(\rho_{\eta^+}^y, \rho_\xi^y) + \angle_y(\rho_\xi^y, \rho_{\eta^-}^y) \geq \angle_x(\eta^+, \xi) + \angle_x(\xi, \eta^-) = \pi$ also by (4.19). Let $\eta \in \Sigma_x X$ denote the (bisector) direction pointing to $(0, -1)$. Then the rays ρ_η^y are orthogonal to $\rho_{\eta^\pm}^y$ and contained in layers $t \times W$.

We choose now $\hat{p}_i$ as a maximum of $d(\sigma_\eta^i, \cdot)$ on the bisector $\{d(\sigma_{\eta^+}^i, \cdot) = d(\sigma_{\eta^-}^i, \cdot)\}$. Again $\hat{p}_i \to x$, and y_0 is a maximum of β_η on $\{\beta_{\eta^+} = \beta_{\eta^-}\}$. If Y is a flat halfplane, then $\{\beta_{\eta^+} = \beta_{\eta^-}\}$ is a halfline and y_0 its endpoint. This is a contradiction because $d(y_0, \cdot)$ has critical points and y_0 cannot lie on the boundary of the halfplane Y. Thus $\dim Y \geq 3$.

The next observation narrows down the possibilities for a 3-dimensional blow-up limit Y.

Lemma 4.21. *If $\dim Y = \dim W + 1 = 3$ and $\dim soul(W) = 1$ (and hence W is a quotient of the flat cylinder), then W must be one-ended.*

Proof. To see this, assume the contrary. Then W splits off a line, i.e. $W \cong \mathbb{R} \times F^1$ with a connected closed 1-orbifold F^1, and we may assume that $y_0 \in 0 \times F^1$. Since y_0 is a maximum of β_η, we have $\beta_\eta(s, f) = -|s|$. There exist two unit speed rays $\rho_i : [0, \infty) \to 0 \times W$ starting from y_0 in antipodal directions, $\angle_y(\dot{\rho}_1(0), \dot{\rho}_2(0)) = \pi$, such that $\beta_\eta(\rho_i(t)) = -t$. From every point in $Y - 0 \times F^1$ starts a unique ray along which β_η decays with slope 1, and thus for $s > 0$ holds $\rho_\eta^{\rho_i(s)}(t) = \rho_i(s + t)$. It follows that there exist points $x_{ij} \in \{d(\sigma_{\eta^+}^i, \cdot) = d(\sigma_{\eta^-}^i, \cdot)\}$ such that, with respect to the rescaled metrics, the segments $x_{ij} \sigma_\eta^i$ converge to the ray ρ_j. In particular, $\delta_i^{-1} d(\hat{p}_i, x_{ij}) \to 0$. On the other hand, without rescaling, the two sequences of segments converge to the same segment $x \sigma_\eta(t_0)$.

It follows by continuity that there exist points $z_{ij} \in x_{ij}\sigma_\eta^i$ such that $d(\sigma_\eta^i, z_{i1}) = d(\sigma_\eta^i, z_{i2})$ and $\tilde{\angle}_{\hat{p}_i}(z_{i1}, z_{i2}) = \frac{\pi}{3}$. We put $l_i = d(z_{i1}, z_{i2})$. Then $d(\hat{p}_i, z_{ij}) \to 0$ and $\delta_i^{-1}d(\hat{p}_i, z_{ij}) \to \infty$. Moreover, $\delta_i^{-1}|d(\hat{p}_i, z_{ij}) - l_i| \to 0$. Let m_i be the midpoints of segments $z_{i1}z_{i2}$.

Triangle comparison applied to the triangles $\Delta(z_{i1}, z_{i2}, \sigma_\eta^i)$ yields $\angle_{z_{ij}}(\sigma_\eta^i, m_i) \gtrsim \frac{\pi}{2}$, and $d(m_i, z_{ij}\sigma_\eta^i) \gtrsim \frac{l_i}{2}$, whereas comparison at $\Delta(z_{i1}, z_{i2}, \hat{p}_i)$ yields $\angle_{z_{ij}}(\hat{p}_i, m_i) \gtrsim \frac{\pi}{3}$ and $d(m_i, z_{ij}\hat{p}_i) \gtrsim \frac{\sqrt{3}}{4}l_i$. It follows that $l_i^{-1}d(m_i, x_{ij}\sigma_\eta^i)$ is bounded away from 0 and $\tilde{\angle}_{\hat{p}_i}(z_{ij}, m_i) \geq \phi_0 > 0$ for large i. The segments $\hat{p}_i m_i$ subconverge to a ray ρ in Y with initial point y_0 and $\angle_{y_0}(\rho_j, \rho) \geq \phi_0$.

Comparison at the triangles $\Delta(x_{ij}, \sigma_\eta^i, \sigma_{\eta\pm}^i)$ yields that $\liminf \tilde{\angle}_{x_{ij}}(z_{ij}, \sigma_{\eta\pm}^i) \geq \frac{\pi}{2}$. Rescaling with the factors $l_i^{-1} \to \infty$ and taking into account that a Gromov-Hausdorff limit splits off a line shows that in fact $\tilde{\angle}_{x_{ij}}(z_{ij}, \sigma_{\eta\pm}^i) \to \frac{\pi}{2}$ and furthermore $\tilde{\angle}_{x_{ij}}(m_i, \sigma_{\eta\pm}^i) \to \frac{\pi}{2}$. Thus $\angle_{y_0}(\rho_{\eta\pm}, \rho) = \frac{\pi}{2}$. In view of $\angle_{y_0}(\rho_j, \rho) \geq \phi_0$, this is a contradiction. $\qquad\square$

Combining the discussion of collapse in the various special cases, we obtain:

Proposition 4.22 (Blow-up limits of strictly larger dimension). *If $X = B(x, 1)$ is a flat cone of dimension ≤ 2, then the base points $\hat{p}_i$ can be chosen so that $\dim Y > \dim X$.*

In fact, the arguments in the special cases above only used the conelikeness of X and the geometry of $\Sigma_x X$, and hence the conclusion of 4.22 holds more generally when X is conelike of dimension ≤ 2.

4.4 Strainers

4.4.1 Position relative to the singular locus

A Riemannian orbifold is a local Alexandrov space with very special singularity structure. The existence of a strainer in a point implies a certain regularity. More precisely, for sufficiently small $\theta > 0$, a point admits a θ-straight m-strainer if and only if its link splits off a join factor isometric to the $(m-1)$-dimensional unit sphere, i.e. if and only if the singular stratum containing it has dimension $\geq m$. The strainer must be almost tangent to the singular stratum.

Thus an interior point in a Riemannian 3-orbifold O admits 3-strainers if and only if it is regular, admits 2-strainers if and only if it is regular or a reflector boundary point, and admits 1-strainers if and only if it is no singular vertex. For instance, $S_{\theta,\mu,-b^2} \cap O^{(0)} = \emptyset$.

4.4.2 Gradient-like vector fields

We recall that for a point $p \in O$ the distance function $d(p, \cdot)$ has directional derivatives. The derivative $\partial_v d(p, \cdot)$ in the direction of a unit tangent vector $v \in \Sigma_x O$ equals $-\cos(v, D_{x,p})$ where $D_{x,p} \subset \Sigma_x O$ is the compact subset of the directions of all segments xp. The function $v \mapsto \partial_v d(p, \cdot)$ on the unit tangent bundle of O is lower semicontinuous outside $\Sigma_p O$, and there its suplevel sets are open. (This argument also works for the distance function from a compact subset of O, e.g. from a component of ∂O if O has boundary.)

For $x \neq p$ and $c \in [0, 1]$ the subset $\{v \in \Sigma_x O : \partial_v d(p, \cdot) \geq c\}$ is totally convex and has diameter $\leq 2 \arccos c$ (e.g. because it has angular distance $\geq \pi - \arccos c$ from $D_{x,p}$). Such a

subset for $c > 0$ can be nonempty only if $x \notin O^{(0)}$, because this requires that $\mathrm{diam}(\Sigma_x O) \geq \pi - \arccos c > \frac{\pi}{2}$. If it is nonempty for a singular point $x \in O^{(1)} \cup O^{(2)}$, then it contains a singular direction at x.

By a standard construction using a partition of unity it follows that for $c \in [0, 1)$ there exists a smooth vector field X on the open subset $\{x \neq p : \exists v \in \Sigma_x O \text{ with } \partial_v d(p, \cdot) > c\}$ which is tangent to the singular locus and satisfies $\partial_X d(p, \cdot) > c$. One calls such a vector field *gradient-like* for $d(p, \cdot)$.

For distinct points a and b there exist gradient-like vector fields for $d(a, \cdot)$ and $d(b, \cdot)$ on the open set $S_{a,b} = \{\angle.(a, b) > \frac{\pi}{2}\} \subset O - \{a, b\}$. (Note that the function $\angle.(a, b)$ is lower semicontinuous on $O - \{a, b\}$.) More precisely, for $\phi \in (0, \frac{\pi}{2}]$ exists a gradient-like vector field X for $d(a, \cdot)$ on $\{\angle.(a, b) > \pi - \phi\}$ with $\angle.(a, X) > \pi - \phi$. Such a vector field satisfies $\angle.(b, -X) \geq \angle.(a, b) - \angle.(a, -X) > \pi - 2\phi$ and $\angle.(b, X) < 2\phi$. (Note that $-X$ is defined, since X is tangent to the singular locus.) Thus, if $\phi \leq \frac{\pi}{4}$ then $\partial_X d(b, \cdot) < 0$ and $\partial_X f_{a,b} > 0$, i.e. X is gradient-like for $f_{a,b}$.

If (a, b) is a θ-straight 1-strainer at p for sufficiently small θ, then its cross section $\Sigma_{x;a,b}$ is near p a topological 2-suborbifold, because a gradient-like vector field for $f_{a,b}$ has local cross sections through p which are smooth 2-suborbifolds, and any two local cross sections can be isotoped to each other using the flow.

4.4.3 Local bilipschitz charts and fibrations by cross sections

Suppose that $(a_1, b_1, a_2, b_2, a_3, b_3)$ is a $< \theta$-straight 3-strainer at a regular point $x \in O$ for some small $\theta > 0$. Then there exist unit vectors $v_1, v_2, v_3 \in \Sigma_x O$ with $\angle_x(a_i, v_i) > \pi - \theta$. It follows that $\angle_x(b_i, -v_i) > \pi - 2\theta$, $|\angle_x(a_i, v_j) - \frac{\pi}{2}|, |\angle_x(b_i, v_j) - \frac{\pi}{2}| < 2\theta$ and $|\angle_x(v_i, v_j) - \frac{\pi}{2}| < 3\theta$ for $i \neq j$. Let X_i be arbitrary commuting smooth vector fields near x with $X_i(x) = v_i$. By continuity, on a sufficiently small neighborhood of x they have length ≈ 1 and satisfy the same angle inequalities $\angle.(a_i, X_i) > \pi - \theta$ and their implications. Thus, they are almost orthogonal, $|\angle(X_i, X_j) - \frac{\pi}{2}| < 3\theta$ for $i \neq j$, and gradient-like for the functions f_{a_i, b_i} associated to the 1-substrainers, $\partial_{X_i} f_{a_i, b_i} > \frac{1}{2}(\cos \phi + \cos 2\phi) > 1 - c\theta^2$, and $|\partial_{X_j} f_{a_i, b_i}| < \sin 2\theta < 2\theta$ for $i \neq j$. The X_i are the coordinate vector fields for some local coordinates, and it follows that $(f_{a_1, b_1}, f_{a_2, b_2}, f_{a_3, b_3})$ restricts to a bilipschitz homeomorphism from a neighborhood of x onto an open subset of $\mathbb{R}^3$. (Compare the discussion in [BGP92, §11.8] und [MT08, 2.4.1].)

An analogous argument can be carried out for a $2\frac{1}{2}$-strainer $(a_1, b_1, a_2, b_2, a_3)$ at a reflector boundary point $x \in O^{(2)}$. Then the 2-substrainer (a_1, b_1, a_2, b_2) is almost tangent to the reflector boundary. One uses an orbifold chart at x, lifts the functions $f_{a_1, b_1}, f_{a_2, b_2}$ and constructs f_{a_3, b_3} by lifting a_3 to a 1-strainer in the chart. Furthermore, one adapts the smooth vector fields X_i to the reflection ι on the chart, i.e. constructs them so that $\iota^* X_1 = X_1$, $\iota^* X_2 = X_2$ and $\iota^* X_3 = -X_3$. One obtains a local bilipschitz homeomorphism to the 3-dimensional halfspace with reflector boundary.

Consider now a $< \theta$-straight 2-strainer (a_1, b_1, a_2, b_2) at a point x. That its cross section $\Sigma_{x;a_1, b_1, a_2, b_2}$ is near x a bilipschitz 1-suborbifold, can be seen as follows. If x is a regular point, then one can choose an $\approx \theta$-straight 1-strainer (a', b') contained in $\Sigma_{x;a_1, b_1, a_2, b_2}$ and, with respect to the local bilipschitz coordinates near x provided by the 3-strainer $(a_1, b_1, a_2, b_2, a', b')$, the

cross section $\Sigma_{x;a_1,b_1,a_2,b_2}$ is a coordinate line. If x is singular, then it must be a reflector boundary point. One chooses a point $a' \in \Sigma_{x;a_1,b_1,a_2,b_2}$ near x and works with the bilipschitz coordinates provided by the $2\frac{1}{2}$-strainer (a_1, b_1, a_2, b_2, a').

Suppose that C is a compact connected component of $\Sigma_{x;a_1,b_1,a_2,b_2}$ such that (a_1, b_1, a_2, b_2) is a $c\theta$-straight 2-strainer at all points of C. Then C is a closed 1-suborbifold and hence homeomorphic to S^1 or the mirrored interval I^1. The map $(f_{a_1,b_1}, f_{a_2,b_2})$ yields a product fibration of a neighborhood of C by cross sections of the 2-strainer. This can be seen as follows using the bilipschitz coordinates near the points $y \in C$. The distance functions f_{a_1,b_1}, f_{a_2,b_2} given by the 2-strainer and the auxiliary distance function f_{a_3,b_3} near y are normalized so that $f_{a_i,b_i}(y) = 0$. For sufficiently small $\epsilon_i > 0$ the map $(f_{a_1,b_1}, f_{a_2,b_2})$ yields near y a product fibration of the box $\{|f_{a_i,b_i}| \le \epsilon_i \, \forall i\}$ over the rectangle $[-\epsilon_1, \epsilon_1] \times [-\epsilon_2, \epsilon_2]$. By covering C with finitely many such boxes one obtains the fibration of a neighborhood.

In the above discussion, one or both of the functions f_{a_1,b_1} and f_{a_2,b_2} can be replaced by $d(a_i, \cdot)$, because their directional derivatives differ only slightly. (Recall that $f_{a_i,b_i} - d(a_i, \cdot)$ is $c'\theta$-Lipschitz in the region where (a_i, b_i) is a $c\theta$-straight 1-strainer, cf. section 3.4.1).

4.5   A decomposition according to the coarse stratification

We start by formulating the collapse assumption on our orbifolds (O, g) needed in this section and the quantities involved in it.

The parameter θ (straightness of 1-strainers) is required to be small, $\theta \in (0, \theta_0]$, where θ_0 is sufficiently small for the arguments in sections 3.4 and 3.5 to apply (cf. the discussion at the beginning of section 3.4). The upper bound for θ will be decreased several times during our later arguments. The parameter μ (accuracy of conical approximation) needs to be sufficiently small so that the conclusions of 3.6 regarding the existence of θ-straight 1-strainers apply, $\mu \le \mu_0(\theta)$ with the constant $\mu_0(\theta)$ from there. The parameter μ determines (together with the fixed parameter σ) via 4.14 the bound $s_1(\sigma, \mu)$ and the scale $\hat{s}_{\mu,-b^2} := \frac{1}{11} s_1(\sigma, \mu) \rho_{-b^2}$. Conical approximation in all points $x \in O$ on scales $s(x) \in [s_1(\sigma, \mu)\rho_{-b^2}(x), \sigma\rho_{-b^2}(x)]$ holds if (O, g) is $(v(\sigma, \mu), -b^2)$-collapsed with the constant $v(\sigma, \mu) > 0$ from 4.14.

In order to make the results on edgy points in section 3.5 available, we need to rule out the existence of $\bar{\theta}_{2\frac{1}{2}}$-straight $2\frac{1}{2}$-strainers with length $\lambda(\theta)\hat{s}_{\mu,-b^2}(x)$ at all points x for a constant $\lambda(\theta)$ which is sufficiently small so that 3.27, 3.28 and 3.29 hold. This is achieved by requiring (O, g) to be $(v(\theta, \mu), -b^2)$-collapsed for a suitable small constant $v(\theta, \mu) > 0$, cf. 4.5. We make it so small that it is smaller than the constant $v(\sigma, \mu)$ mentioned above.

Furthermore, in order to obtain sufficient collapse on the scale $\theta^4 \hat{s}_{\mu,-b^2}(x)$ for small θ, we ask that $v(\theta, \mu) \le (\theta^5 s_1(\sigma, \mu))^3$. We will also assume that (O, g) has $(v(\theta, \mu), s_0, K)$-curvature control below scale ρ_{-b^2}.

4.5.1   The 2-strained region

We define the *2-strained region* $R_{\theta,\mu,-b^2} \subset O$ as the open set consisting of all points x which admit $< C_1\theta$-straight 2-strainers of length $> \theta^4 \hat{s}_{\mu,-b^2}(x)$, where C_1 is the constant from 3.21. Note that $R_{\theta,\mu,-b^2} \cap O^{(sing)} \subset O^{(2)}$.

We will show that for sufficiently small θ the 2-strained region admits metrically an almost product fibration with short fibers almost orthogonal to the 2-strainers. (The smallness of the parameter μ is not important at this point, because we are not yet using conical approximation from 4.14.)

We begin with a local approximation result:

Proposition 4.23. *For $\epsilon > 0$ exists $\theta_1 = \theta_1(\epsilon) > 0$ such that:*

Let $\theta \leq \theta_1$ and $\mu \leq \mu_0(\theta)$. If (O, g) is $(v(\theta, \mu), -b^2)$-collapsed with $(v(\theta, \mu), s_0, K)$-curvature control below scale ρ_{-b^2}, and if (a_1, b_1, a_2, b_2) is a $C_1\theta$-straight $\theta^4 \hat{s}_{\mu, -b^2}(x)$-long 2-strainer at x, then $\text{diam}(\Sigma^o_{x;a_1,b_1,a_2,b_2})^{-1} \cdot (O, x)$ is ϵ-close in the pointed $\mathcal{C}^5$-topology to the product $\mathbb{R}^2 \times F^1$ of the euclidean plane with a connected closed 1-orbifold of diameter 1.

Proof. Let $-b_i^2 \in [-1, 0)$ and $\theta_i, \mu_i > 0$ such that $\theta_i \to 0$ and $\mu_i \leq \mu_0(\theta_i)$. Suppose that the orbifolds (O_i, g_i) are $(v(\theta_i, \mu_i), -b_i^2)$-collapsed with $(v(\theta_i, \mu_i), s_0, K)$-curvature control below the scales $\rho_{-b_i^2}$, and that the $(a_1^i, b_1^i, a_2^i, b_2^i)$ are $C_1\theta_i$-straight $\theta_i^4 \hat{s}_{\mu_i, -b_i^2}(x_i)$-long 2-strainers at the points x_i. We have to show that the conlusion of the proposition holds for large i.

Consider the cross sections $\Sigma_i^o = \Sigma^o_{(x_i;a_1^i,b_1^i,a_2^i,b_2^i)}$. For any point $x_i \neq y_i \in \Sigma_i^o$, the $2\frac{1}{2}$-strainer $(a_1^i, b_1^i, a_2^i, b_2^i, y_i)$ at x_i is $\bar{\theta}_{2\frac{1}{2}}$-straight for large i. Since $v(\theta, \mu) \leq (\theta^5 s_1(\sigma, \mu))^3$, it follows that $\lambda_i := (\theta_i^4 \hat{s}_{\mu_i, -b_i^2}(x_i))^{-1} \text{diam}(\Sigma_i^o) \to 0$, cf. 4.5. The rescaled pointed orbifolds $(\lambda_i \theta_i^4 \hat{s}_{\mu_i, -b_i^2}(x_i))^{-1} \cdot (O_i, x_i)$ Gromov-Hausdorff subconverge to a pointed Alexandrov space (X, x_0) with dimension ≤ 3 and curvature ≥ 0. Moreover, the broken segments $a_1^i x_i b_1^i$ and $a_2^i x_i b_2^i$ (sub)converge to two perpendicular lines through x_0, and hence X splits metrically as a product $\mathbb{R}^2 \times \Sigma$.

To see that the rescaled cross sections $(\lambda_i \theta_i^4 \hat{s}_{\mu_i, -b_i^2}(x_i))^{-1} \cdot \Sigma_i^o$ subconverge to Σ, we note that $(\lambda_i \theta_i^4 \hat{s}_{\mu_i, -b_i^2}(x_i))^{-1}(d(a_j^i, \cdot) - d(a_j^i, x_i))$ and $(\lambda_i \theta_i^4 \hat{s}_{\mu_i, -b_i^2}(x_i))^{-1}(d(b_j^i, \cdot) - d(b_j^i, x_i))$ subconverge (due to Arzela-Ascoli) to concave 1-Lipschitz functions α_j and β_j with $\alpha_j(x_0) = \beta_j(x_0) = 0$. The concavity of the sums $\alpha_j + \beta_j$ implies together with the triangle inequality that the functions α_j and β_j are constant on fibers $pt \times \Sigma$. Since for any sequence points $y_i \in \Sigma_i^o$ holds that $(\lambda_i \theta_i^4 \hat{s}_{\mu_i, -b_i^2}(x_i))^{-1}(d(a_j^i, y_i) - d(a_j^i, x_i)), (\lambda_i \theta_i^4 \hat{s}_{\mu_i, -b_i^2}(x_i))^{-1}(d(b_j^i, y_i) - d(b_j^i, x_i)) \to 0$, compare (3.13), it follows that $\Sigma_i^o \to \Sigma$. Therefore Σ is a compact Alexandrov space with diameter 1 and dimension 1.

Since in the blow-up limit $(\lambda_i \theta_i^4 \hat{s}_{\mu_i, -b_i^2}(x_i))^{-1} \cdot (O_i, x_i) \to (X, x_0)$ we have no dimension drop, after passing to another subsequence, the convergence can be improved to $\mathcal{C}^5$-smooth convergence and $F^1 = \Sigma$ is a connected closed 1-orbifold, cf. 4.7. $\qquad\square$

Note that in the situation of the proposition, $\text{diam}(\Sigma_{x;a_1,b_1,a_2,b_2}) << \theta^4 \hat{s}_{\mu, -b^2}(x)$. Moreover, for $x \in R_{\theta, \mu, -b^2}$ the cross sections $\Sigma^o_{x;a_1,b_1,a_2,b_2}$ of different $C_1\theta$-straight $\theta^4 \hat{s}_{\mu, -b^2}(x)$-long 2-strainers (a_1, b_1, a_2, b_2) at x have Hausdorff distance $<< \text{diam}(\Sigma^o_{x;a_1,b_1,a_2,b_2})$ and almost equal diameters, and we define the *width* $w(x)$ at x as the infimum of these diameters.

The fiber direction of a local approximation as in 4.23 yields a smooth line field which is *almost vertical* in the sense that it is perpendicular to the stratum $O^{(2)}$ and almost perpendicular to sufficiently long segments. Any two such local line fields almost agree on the overlaps of their domains of definition, and using a partition of unity we can combine such local line fields to a global almost vertical line field $L = L_{\theta, \mu, -b^2}$ on $R_{\theta, \mu, -b^2}$. More precisely, for (small) $\nu > 0$ exists $\theta_1 = \theta_1(\nu) > 0$ such that the following holds: If $\theta \leq \theta_1$, $\mu \leq \mu_0(\theta)$ and if (O, g) is $(v(\theta, \mu), -b^2)$-

collapsed with $(v(\theta,\mu), s_0, K)$-curvature control below scale ρ_{-b^2}, then for $x \in R_{\theta,\mu,-b^2}$ the line $L(x)$ has angle $> \frac{\pi}{2} - \nu$ with any segment of length $10\nu^{-1}w(x)$ initiating in x, and in particular with any segment of length $\theta^4 \hat{s}_{\mu,-b^2}(x) >> w(x)$. In fact, a line field $L_{\theta,\mu,-b^2}$ with these properties can be constructed on the slightly larger open set $\hat{R}_{\theta,\mu,-b^2} := \cup_{x \in R_{\theta,\mu,-b^2}} B(x, \frac{1}{\nu}w(x))$. The reflector boundary $O^{(2)}$ is, where it meets $R_{\theta,\mu,-b^2}$, *almost horizontal* in the sense that it is almost tangent to sufficiently long segments and, in particular, to sufficiently long 2-strainers.

The trajectories of L starting in a point $x \in R_{\theta,\mu,-b^2}$ move almost orthogonally to sufficiently long segments and therefore remain close to the cross section $\Sigma^o_{x;a_1,b_1,a_2,b_2}$ of a suitable 2-strainer as above for length at least $>> w(x)$.

If $x \in O^{(2)} \cap R_{\theta,\mu,-b^2}$, then the trajectory is orthogonal to $O^{(2)}$ in x and reaches $O^{(2)}$ again after length $\approx w(x)$. More generally, all trajectories intersecting $B(x, 10w(x))$ have length $\approx w(x)$ and connect reflector boundary points. The construction of an almost product fibration by mirrored intervals close to $O^{(2)} \cap R_{\theta,\mu,-b^2}$ is therefore immediate.

Away from the reflector boundary, the L-trajectories starting in points $x \in R_{\theta,\mu,-b^2}$ almost close up after length $\approx 2w(x)$. The question whether L can be globally perturbed to an integrable line field with closed trajectories of lengths $\approx 2w(x)$ has been treated in [MT08, §4.2] in a very similar setting. The discussion there uses only the control on finitely many derivatives of the curvature tensor and goes through without change in the situation considered here. One obtains the following result which can be considered as a version of a special case of Yamaguchi's Fibration Theorem [Ya91] "without a priori given base".

Proposition 4.24 (Almost vertical fibration of the 2-strained region, cf. [MT08, Prop. 4.4]). *For $\nu > 0$ exists $\theta_1 = \theta_1(\nu) > 0$ such that:*

If $\theta \leq \theta_1$, $\mu \leq \mu_0(\theta)$ and if (O, g) is $(v(\theta,\mu), -b^2)$-collapsed with $(v(\theta,\mu), s_0, K)$-curvature control below scale ρ_{-b^2}, then there exists an open subset U, $R_{\theta,\mu,-b^2} \subseteq U \subseteq \hat{R}_{\theta,\mu,-b^2}$, such that every connected component of U is the total space of a smooth orbifold fibration with fiber S^1 or the mirrored interval I^1, and all fibers have angle $< \nu$ with the almost vertical line field $L_{\theta,\mu,-b^2}$.

The local fibrations provided by the product approximations in 4.23 have only a finite degree of regularity, but the global fibration of U obtained by interpolating these local fibrations can be smoothed. The smoothness will however not be important to us.

From now on, we fix some small positive value of ν (e.g. $\nu = \frac{1}{2010}$) and set $\theta_1 = \theta_1(\nu)$. Moreover, whenever an orbifold is sufficiently collapsed, we implicitly fix a fibration as in 4.24.

4.5.2 Edges

Definition 4.25. We define a point $x \in S_{\theta,\mu,-b^2}$ to be $(\theta, \mu, -b^2)$-*edgy* relative to an equilateral $< \theta$-straight 1-strainer (a, b) at x with length in $(\hat{s}_{\mu,-b^2}(x), \frac{3}{2}\hat{s}_{\mu,-b^2}(x))$ if $\mathrm{diam}(\Sigma^o_{x;a,b}) > \theta^{\frac{5}{2}} \hat{s}_{\mu,-b^2}(x)$ and if $\Sigma_{x;a,b}$ contains no $\frac{\pi}{2}$-straight $\theta^4 \hat{s}_{\mu,-b^2}(x)$-long 1-strainers at x, compare 3.25.

Let us briefly discuss the correspondence between the two definitions 3.25 and 4.25. First, we recall from our discussion in section 3.1.3 that a $(\theta, \mu, -b^2)$-edgy point x relative to a $< \theta$-straight 1-strainer (a, b) with length in $(\hat{s}_{\mu,-b^2}(x), \frac{3}{2}\hat{s}_{\mu,-b^2}(x))$ is also θ-edgy in the space $(\hat{s}_{\mu,-b^2}(x))^{-1} \cdot B(x, \rho_{-b^2}(x))$ of curvature ≥ -1 in the sense of Definition 3.25: In the rescaled

space, the strainer (a, b) is $< 2\theta$-straight of length $> (1 - \theta)$ and the cross section $\Sigma_{x;a,b}$ cannot contain any $\frac{\pi}{2}$-straight 1-strainer of length θ^4.

Conversely, suppose that an equilateral 1-strainer (a, b) is $< \theta$-straight at x with length in $(\hat{s}_{\mu,-b^2}(x), \frac{3}{2}\hat{s}_{\mu,-b^2}(x))$ and that in the rescaled space $\hat{s}_{\mu,-b^2}(x)^{-1} \cdot B(x, \rho_{-b^2}(x))$ x is θ-edgy relative to (a, b) (again in the sense of Definition 3.25). Then Remark 3.22 together with the discussion at the end of 3.1.3 implies that x is also $(\theta, \mu, -b^2)$-edgy. For suppose that $\Sigma_{x;a,b}$ contains a $\frac{\pi}{2}$-straight $\theta^4 \hat{s}_{\mu,-b^2}(x)$-long 1-strainer at x. Then after rescaling this strainer is still $< \frac{3\pi}{4}$-straight and hence in fact $< C_1\theta$-straight which is a contradiction.

By construction, for any $x \in O$ there are no $\bar{\theta}_{2\frac{1}{2}}$-straight $2\frac{1}{2}$-strainers of length $\lambda(\theta)$ in the rescaled space $\hat{s}_{\mu,-b^2}(x)^{-1}B(x, \frac{1}{2}\rho_{-b^2}(x))$ Moreover, estimate 4.3 implies that on the ball $B(x, \theta\hat{s}_{\mu,-b^2}(x))$ we have the estimate $\hat{s}_{\mu,-b^2}/\hat{s}_{\mu,-b^2}(x) \in (1 - \theta, 1 + \theta)$.

This allows us to generalize the above arguments to the following results: If $y \in B(\theta\hat{s}_{\mu,-b^2}(x))$ is $(\theta, \mu, -b^2)$-edgy relative to a 1-strainer (a, b) then it is θ-weakly in the rescaled space $\hat{s}_{\mu,-b^2}(x)^{-1} \cdot B(x, \rho_{-b^2}(x))$ of curvature ≥ -1. Conversely, suppose that $y \in B(\theta\hat{s}_{\mu,-b^2}(x))$ admits an equilateral $< \theta$-straight 1-strainer (a, b) with length in $(\hat{s}_{\mu,-b^2}(y), \frac{3}{2}\hat{s}_{\mu,-b^2}(y))$ such that y is θ-strongly edgy relative to (a, b) in $\hat{s}_{\mu,-b^2}(x)^{-1} \cdot B(x, \rho_{-b^2}(x))$. Then y is also $(\theta, \mu, -b^2)$-edgy.

Hence our results from section 3.5.2 can be applied to our present situation and used to control e.g. the relative position of edgy points.

We will globally construct tubes along the "coarse edges". Our previous discussion applies provided that θ and μ are sufficiently small (i.e. $\theta \leq \theta_1$ and $\mu \leq \mu_0(\theta)$), and that (O, g) is $(v(\theta, \mu), -b^2)$-collapsed with $(v(\theta, \mu), s_0, K)$-curvature control below scale ρ_{-b^2}.

For every $(\theta, \mu, -b^2)$-edgy point $x \in O$, let (a_x, b_x) be some equilateral $< \theta$-straight 1-strainer with length in $(\hat{s}_{\mu,-b^2}(x), \frac{3}{2}\hat{s}_{\mu,-b^2}(x))$ relative to which x is edgy. (Since 1-strainers at x are almost unique by 3.23, it does not matter which one we use.) Associated to it is the truncated cross section $\check{\Sigma}_x = \Sigma_{x;a_x,b_x} \cap B(x, \frac{1}{2}\theta^3 \hat{s}_{\mu,-b^2}(x))$.

For $j = 1, 2, 3$ we consider the fibers $\gamma_x^j = \Sigma_{x;a_x,b_x} \cap B(x, \frac{j}{8}\theta^3 \hat{s}_{\mu,-b^2}(x))$ of the partial (topological) product fibration of $\Sigma_{x;a_x,b_x}$ induced by $d(x, \cdot)$. (Compare the discussion in section 4.4.3.) The γ_x^j and also their $\frac{1}{16}\theta^3 \hat{s}_{\mu,-b^2}(x)$-neighborhoods are contained in the 2-strained region $R_{\theta,\mu,-b^2}$, and they are almost vertical in the sense that they are isotopic to a fiber of the fibration given by 4.24 by a small isotopy, say, supported on the $\theta^4 \hat{s}_{\mu,-b^2}(x)$-neighborhood of γ_x^j.

An almost unique 1-strainer at a $(\theta, \mu, -b^2)$-edgy point x locally defines a vector field on the ball $B(x, \theta\hat{s}_{\mu,-b^2})$ as in section 4.4.2. By interpolating these fields by a partition of unity, we obtain a smooth vector field $L = L_{edge}$ tangent to the singular locus whose open domain of definition contains the balls $B(x, \theta\hat{s}_{\mu,-b^2}(x))$ around the $(\theta, \mu, -b^2)$-edgy points x, and which is on these balls almost parallel to the 1-strainers (a_x, b_x), meaning that $\angle.(L_{edge}, a_x), \angle.(L_{edge}, b_x) \notin [c\theta^{\frac{1}{2}}, \pi - c\theta^{\frac{1}{2}}]$ on $B(x, \theta\hat{s}_{\mu,-b^2}(x))$, see 3.27. In particular, we have $|\angle.(L_{edge}, x) - \frac{\pi}{2}| < c'\theta^{\frac{1}{2}}$ on the $\frac{1}{7}\theta^3 \hat{s}_{\mu,-b^2}(x)$-neighborhood of γ_x^2.

We choose a maximal subfamily of pairs $(x, \check{\Sigma}_x)$ such that the corresponding subset ϵ of $(\theta, \mu, -b^2)$-edgy points x is separated in the sense that for any two distinct points $x_1, x_2 \in \epsilon$ holds $d(x_1, x_2) > \theta^{\frac{10}{3}}\hat{s}_{\mu,-b^2}(x_1)$. By 3.29, the $\check{\Sigma}_x$ for $x \in \epsilon$ are pairwise disjoint.

We call $x_1, x_2 \in \epsilon$ *adjacent*, if $d(x_1, x_2) < 4\theta^{\frac{10}{3}}\hat{s}_{\mu,-b^2}(x_1)$ and if the arc in the L_{edge}-trajectory (leaf) connecting x_1 to its intersection point with $\check{\Sigma}_{x_2}$ does not meet the other cross sections $(x, \check{\Sigma}_x)$ for $x \in \epsilon - \{x_1, x_2\}$. By 3.28, we have $d(x_1, x_2) \approx \theta^{\frac{10}{3}}\hat{s}_{\mu,-b^2}(x_1)$ unless points on the segment $x_1 x_2$ have no θ-straight $\hat{s}_{\mu,-b^2}$-long 1-strainers or have such strainers with cross sections of diameter $\approx \theta^{\frac{5}{2}}$. The relation of adjacency generates an equivalence relation on ϵ, and we call an equivalence class a *chain* of $(\theta, \mu, -b^2)$-edgy points. Each chain can be given a linear or cyclic order.

Consider two adjacent edgy points $x_1, x_2 \in \epsilon$. The cross sections $\check{\Sigma}_{x_i}$ have Hausdorff distance $< \theta^{\frac{99}{30}}\hat{s}_{\mu,-b^2}(x_1)$ by 3.29. Using the integral curves of the line field L_{edge}, we flow the 1-orbifolds $\gamma_{x_1}^j$ from $\check{\Sigma}_{x_1}$ into $\check{\Sigma}_{x_2}$. Their images $\gamma_{x_1,x_2}'^j$ in $\check{\Sigma}_{x_2}$ are $< \theta^{\frac{9}{20}+\frac{99}{30}}\hat{s}_{\mu,-b^2}(x_1) < \theta^{\frac{7}{2}}\hat{s}_{\mu,-b^2}(x_1)$-close to the $\gamma_{x_2}^j$, compare the discussion of the maps $\Phi_t^{a,b}$ in section 3.4.1. Moreover, since the isotopy of $\gamma_{x_1}^j$ to $\gamma_{x_1,x_2}'^j$ takes place inside a $\theta^{\frac{98}{30}}\hat{s}_{\mu,-b^2}(x_1)$-ball contained in $R_{\theta,\mu,-b^2}$, $\gamma_{x_1,x_2}'^j$ is isotopic by a small isotopy to the almost vertical fibers of the fibration of $R_{\theta,\mu,-b^2}$, and therefore it must inside $\check{\Sigma}_{x_2}$ be homotopic to $\gamma_{x_2}^j$, and hence isotopic by a small isotopy. Thus the trace of the isotopy of $\gamma_{x_1}^j$ to $\gamma_{x_1,x_2}'^j$ can be adjusted (by a small isotopy supported near $\gamma_{x_2}^j$) to a 2-suborbifold S_{x_1,x_2}^j homeomorphic to $\cong \gamma_{x_1}^j \times [0,1]$, contained in $A(x; (\frac{j}{8} - \frac{1}{100})\theta^3 \hat{s}_{\mu,-b^2}(x), (\frac{j}{8} + \frac{1}{100})\theta^3 \hat{s}_{\mu,-b^2}(x))$, lying between $\check{\Sigma}_{x_1}$ and $\check{\Sigma}_{x_2}$, and with boundary $\gamma_{x_1}^j \cup \gamma_{x_2}^j$. By concatenating the S_{x_1,x_2}^j, we obtain three disjoint embedded 2-suborbifolds S^j following along the chains of edgy points. We call the region contained between S^1 and the two final cross section of the chain a *tube* along the coarse edge.

Note that again by 3.28, a chain can only end inside a $(\theta, \mu, -b^2)$-hump or if the cross sections to $C_0\theta$-straight $\hat{s}_{\mu,-b^2}$-long 1-strainers have diameter $\approx \theta^{\frac{5}{2}}$. It is also possible that an edgy point has no adjacent edgy points. We simply discard such isolated cross sections.

The simplest *interface* between chains of edgy points (the "coarse edge") and the rest of O arises for a *cyclic chain* $\kappa \subseteq \epsilon$ of $(\theta, \mu, -b^2)$-edgy points. The parts $S_\kappa^j \subset S^j$ corresponding to κ are then closed 2-suborbifolds. Let T_κ^j denote the tube containing κ and bounded by S_κ^j, and let $A_\kappa^{i,j} := T_\kappa^j - int(T_\kappa^i)$ for $1 \le i < j \le 3$. The T_κ^j and $A_\kappa^{i,j}$ are compact 3-suborbifolds compatibly fibering over the circle, and $A_\kappa^{1,3} \subset R_{\theta,\mu,-b^2}$. According to 4.28, the fiber of T_κ^j is a compact 2-orbifold with Euler characteristic $\chi \ge 0$ and one boundary component. Note that S_κ^2 separates S_κ^1 and S_κ^3.

We wish to replace S_κ^2 by a 2-suborbifold which is *vertically saturated*, i.e. saturated with respect to the fibration of $R_{\theta,\mu,-b^2}$, cf. 4.24. To do so, we take a vertically saturated compact connected 3-suborbifold W such that $S_\kappa^2 \subset int(W)$ and $W \subset int(A_\kappa^{1,3})$. Then W separates S_κ^1 and S_κ^3, and according to Lemma 4.26 below, one of the boundary components of W separates S_κ^1 and S_κ^3. We denote this boundary component by $S_\kappa^{2,v}$.

Then as a consequence of Lemma 4.27, $S_\kappa^{2,v}$ is isotopic to the S_κ^j. In other words, we can isotope S_κ^2 by an isotopy supported in $int(A_\kappa^{1,3})$ so that it becomes vertically saturated and the fibrations on it induced by $R_{\theta,\mu,-b^2}$ and T_κ^2 match.

Lemma 4.26. *Let Σ be a connected closed 2-orbifold without singular points and let $W \subset \Sigma \times [0,1]$ be a compact connected 3-suborbifold disjoint from $\Sigma \times 0$ and $\Sigma \times 1$ and separating them. Then some component of ∂W separates them also.*

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Proof. For the purpose of this lemma, we consider Σ and Σ' as compact manifolds, possibly with boundary.

Based on the existence and uniqueness of smooth structures on 3-manifolds and the uniqueness up to isotopy of smooth structures on 2-manifolds, we know that there exists a smooth structure on $\Sigma \times [0, 1]$ with respect to which the embedded topological 2-submanifold ∂W is a smooth submanifold. (Cut along ∂W, put a smooth structure and glue again after adjusting the induced smooth structures on the boundaries by an isotopy in a collar. See e.g. [Mu60], [Wh61], [Ep66].)

Moreover, given a smooth structure on Σ (and hence on $\Sigma \times [0, 1]$), there exists a homeomorphism of $\Sigma \times [0, 1]$ carrying the embedded topological 2-submanifolds ∂W to a smooth submanifold. Hence we may work without loss of generality in the smooth category.

Let V denote the component of $\Sigma \times (0, 1) - int(W)$ containing $\Sigma \times 0$. Then $\Sigma' = V \cap W$ separates $\Sigma \times 0$ and $\Sigma \times 1$, and we have to show that it is connected. Suppose the contrary, i.e. that it decomposes as the disjoint union $\Sigma' = \Sigma'_1 \cup \Sigma'_2$ of closed 2-submanifolds. Then there exists an embedded circle γ in $\Sigma \times (0, 1)$ which intersects Σ'_1 once transversally. This is absurd because γ can be homotoped into $\Sigma \times 0$. $\square$

The following fact is for tori a simple special case of a result of Waldhausen [Wa67, 2.8]. The arguments in the non-orientable and orbifold cases are similar.

Lemma 4.27. *(i) Let Σ and Σ' be closed surfaces, each of which is homeomorphic to the 2-torus T^2 or to the Klein bottle K^2. Suppose that $\Sigma' \subset \Sigma \times [0, 1]$ is embedded so that it is disjoint from $\Sigma \times 0$ and $\Sigma \times 1$ and separates them. Then Σ' is isotopic to $\Sigma \times 0$ and $\Sigma \times 1$.*

(ii) The same conclusion holds if Σ and Σ' are closed 2-orbifolds, each of which is homeomorphic to the annulus Ann^2 or the Möbius strip $Möb^2$ with reflector boundary.

Proof. Again we can assume without loss of generality that Σ' is a smooth surface, respectively, suborbifold.

(i) Cut open $\Sigma \times [0, 1]$ along an annulus A so as to obtain a solid torus $[0, 1] \times [0, 1] \times S^1$. After adjusting Σ' we can assume that it intersects A transversally in circles. Because Σ' separates $\Sigma \times [0, 1]$, every circle which is null-homotopic in A bounds a 2-ball in Σ'. (Otherwise, by the orientability and irreducibility of the solid torus, such a circle would decompose Σ' into an annulus, and we could compress Σ' to a circle or a point.) Using irreducibility again, we can isotope Σ' such that it intersects A only in circles which are not null-homotopic and decompose Σ' into annuli. Moreover, we can assume that these circles are vertical in the fibration of the full torus by circles. Hence every annulus component of $\Sigma' - A$ can be isotoped to be vertical, too. (Cf. [Wa67, 2.4].) Thus, Σ' can be isotoped to be vertical in a fibration of $\Sigma \times [0, 1]$ by circles. This clearly implies (i).

(ii) Without loss of generality, we can assume that every boundary component of $\partial \Sigma'$ is horizontal in the product $\Sigma \times [0, 1]$. In the case where Σ is homeomorphic to Ann^2, so is Σ' and the two boundary components of Σ' lie above different boundary components of Σ.

As above, we cut open $\Sigma \times [0, 1]$ along a 2-ball B to obtain a 3-ball $[0, 1] \times [0, 1] \times [0, 1]$. Again, we arrange that Σ' is transversal to B and hence intersects it in null-homotopic circles or in intervals connecting opposite sides of $B \cong [0, 1] \times [0, 1]$ or one side to itself.

Circles in $B \cap \Sigma'$ must again bound 2-balls in Σ' and hence can be removed by suitable isotopies. (If Σ' is a Moebius band, this follows from the orientability of the 3-ball. If Σ' is an annulus which is decomposed into two annuli by a component of $\Sigma' \cap B$, we would obtain a compression disc for Σ which is equally impossible.)

Similarly, it is impossible that an interval component of $\Sigma' \cap B$ connects one side of B to itself: This can only occur if Σ was a Moebius band and Σ' an annulus. However, it follows in this case that Σ' can be compressed to the $\partial\Sigma \times [0,1]$.

This implies that without loss of generality $\Sigma' \cap B$ consists of one or two intervals connecting opposite sides of the square B. Since they decompose Σ' into 2-balls, claim (ii) now follows. $\square$

4.5.3 Necks

Throughout this section, we assume that $\theta < \theta_1$ and $\mu < \mu_0(\theta)$ are chosen sufficiently small, and that (O, g) is $(v(\theta, \mu), -b^2)$-collapsed with $(v(\theta, \mu), s_0, K)$-curvature control below scale ρ_{-b^2}.

We define a point $x \in O$ to be $(\theta, \mu, -b^2)$-*necklike* relative to an equilateral $< \theta$-straight 1-strainer (a, b) at x with length in $(\hat{s}_{\mu,-b^2}(x), \frac{3}{2}\hat{s}_{\mu,-b^2}(x))$ if $\operatorname{diam}(\Sigma^o_{x;a,b}) < \theta^2 \hat{s}_{\mu,-b^2}(x)$, compare 3.30. We call the open subset $N_{\theta,\mu,-b^2} \subseteq S_{\theta,\mu,-b^2}$ of $(\theta, \mu, -b^2)$-necklike points the *necklike region* of O. It does not contain any singular vertices, $O^{(0)} \cap N_{\theta,\mu,-b^2} = \emptyset$.

As in the beginning of the previous section, we verify that a point $x \in O$ is $(\theta, \mu, -b^2)$-necklike relative to a 1-strainer (a, b) if and only if it is θ-necklike in the rescaled space $(\hat{s}_{\mu,-b^2}(x))^{-1} \cdot B(x, \rho_{-b^2}(x))$ in the sense of Definition 3.30.

Similarly, if $y \in B(x, \rho_{-b^2}(x))$ is $(\theta, \mu, -b^2)$-necklike it is θ-weakly necklike relative to (a, b) in $(\hat{s}_{\mu,-b^2}(x))^{-1} \cdot B(x, \rho_{-b^2}(x))$. If y admits an equilateral $< \theta$-straight 1-strainer (a, b) with length in $(\hat{s}_{\mu,-b^2}(y), \frac{3}{2}\hat{s}_{\mu,-b^2}(y))$ and is θ-strongly necklike relative to (a, b) in $(\hat{s}_{\mu,-b^2}(x))^{-1} \cdot B(x, \rho_{-b^2}(x))$, it is also $(\theta, \mu, -b^2)$-necklike.

Thus we can again use the results from 3.5.2 to control the existence and relative position of necklike points.

As in 4.5.2, we construct a smooth line field L_{neck} tangent to the singular locus whose open domain of definition contains the balls $B(x, \theta^{\frac{3}{2}} \hat{s}_{\mu,-b^2}(x))$ around all $(\theta, \mu, -b^2)$-necklike points x, and which is on every such ball almost parallel to the equilateral 1-strainers (a, b) at x with length in $(\hat{s}_{\mu,-b^2}(x), \frac{3}{2}\hat{s}_{\mu,-b^2}(x))$, i.e. $\angle(L_{neck}, a), \angle(L_{neck}, b) < \theta^{\frac{1}{2}}$ on that ball. The line fields L_{neck} and L_{edge} can be matched in the overlap of their domains of definition.

For a $< \theta$-straight equilateral 1-strainer (a, b) with length in $(\hat{s}_{\mu,-b^2}(x), \frac{3}{2}\hat{s}_{\mu,-b^2}(x), \Sigma_{x;a,b}$ is a closed topological 2-suborbifold almost perpendicular to L_{neck}, i.e. it has angle $> \frac{\pi}{2} - c''\theta$ with it. We call $\Sigma_{x;a,b}$ a *neck cross section* through the point x.

If two neck cross sections $\Sigma_{x_1;a_1,b_1}$ and $\Sigma_{x_2;a_2,b_2}$ intersect, then they have Hausdorff distance $< c\theta^3 \hat{s}_{\mu,-b^2}(x)$ by 3.32. If they are disjoint but also not too far apart from each other, say if they have Hausdorff distance $< \theta^{\frac{5}{3}} \hat{s}_{\mu,-b^2}(x)$, then one can move one of the cross sections to the other along the trajectories of L_{neck}, and therefore the $\Sigma^o_{x_i;a_i,b_i}$ are in this case topologically *parallel*, i.e. they bound a product suborbifold $\cong \Sigma^o_{x_i;a_i,b_i} \times [0,1]$.

Among all neck cross sections, we choose a maximal subfamily ν such that any two distinct

cross sections $\Sigma_{x_1;a_1,b_1}$ and $\Sigma_{x_2;a_2,b_2}$ in ν have Hausdorff distance $> \theta^{\frac{5}{2}}\hat{s}_{\mu,-b^2}(x_1)$. In particular, they are disjoint. Due to the compactness of O, ν is finite. Inside ν, we form equivalence classes of topologically parallel cross sections. Each equivalence class has a linear or cyclic order. Any two successive cross sections in it are topologically parallel and bound a cylinder ("segment") homeomorphic to the product of one of them with the compact interval. By concatenating these pieces, the equivalence class yields an embedded *neck* in O which fibers over the interval or over the circle, unless the equivalence class consists only of a single neck cross section. Such an isolated neck cross section has diameter $\approx \theta^2\hat{s}_{\mu,-b^2}(x)$ (with respect to some point x in it); it is contained in the union of the 2-strained and edgy regions, and we simply disregard it. Any two necks are disjoint. The union N of all necks is a compact 3-suborbifold.

A *cyclic* neck is a closed 3-orbifold and hence fills out O entirely. The topology of cyclic necks will be determined later.

A *linear* neck has two boundary components which are neck cross sections. We call an end of the neck *thick* if its boundary $\Sigma_{x;a,b}$ has diameter $> \theta^{\frac{49}{20}}\hat{s}_{\mu,-b^2}(x)$, and *thin* otherwise. If the end is thin, then nearby $\Sigma_{x;a,b}$, according to our construction e.g. at distance $< \theta^{\frac{49}{20}}\hat{s}_{\mu,-b^2}(x)$, must exist points outside the 1-strained region $S_{\theta,\mu,-b^2}$, i.e. a $(\theta,\mu,-b^2)$-hump, cf. 4.15. The interface between a thin end of a neck and a hump will be discussed later.

Let $\Sigma_{x;a,b}$ be a neck cross section with $\mathrm{diam}(\Sigma_{x;a,b}) > \theta^{\frac{49}{20}}\hat{s}_{\mu,-b^2}(x)$. Then every point in $\Sigma_{x;a,b}$ is $(\theta,\mu,-b^2)$-edgy or belongs to the 2-strained region $R_{\theta,\mu,-b^2}$.

Let $y \in \Sigma_{x;a,b} \cap R_{\theta,\mu,-b^2}$. Then $\Sigma_{x;a,b}$ contains a $C_1\theta$-straight 1-strainer (z_1,z_2) at y with length $\theta^4\hat{s}_{\mu,-b^2}(y)$, cf. 3.20 and 3.21(i). As discussed in section 4.4.3, the portion $A_y = \Sigma_{x;a,b} \cap \{|f_{z_1,z_2}| \leq \frac{1}{10}\theta^4\hat{s}_{\mu,-b^2}(y)\} \cap B(y,\theta^4\hat{s}_{\mu,-b^2}(y))$ of the cross section fibers over a compact interval with fibers the f_{z_1,z_2}-level sets. These are embedded 1-suborbifolds $\cong S^1$ or I^1. Let $\gamma_y = \Sigma_{x;a,b} \cap f_{z_1,z_2}^{-1}(0) = \Sigma^o_{y;a,b,z_1,z_2}$ denote the central fiber.

To combine these local fibrations to a global one, we choose a maximal family F of γ_y's so that any two distinct $\gamma_{y_1}, \gamma_{y_2} \in F$ have Hausdorff distance $> \frac{1}{100}\theta^4\hat{s}_{\mu,-b^2}(y_1)$. The family F is finite, since $\Sigma_{x;a,b}$ is compact. If γ_{y_1} and γ_{y_2} have Hausdorff distance $< \frac{9}{100}\theta^4\hat{s}_{\mu,-b^2}(y_1)$, then γ_{y_2} separates A_{y_1} and is isotopic inside A_{y_1} (by a small isotopy) to a fiber of the above fibration of A_{y_1}. We call γ_{y_1} and γ_{y_2} *adjacent*, if they are not separated inside A_{y_1} by another $\gamma_y \in F$. In this case, they have Hausdorff distance $\approx \frac{1}{100}\theta^4\hat{s}_{\mu,-b^2}(y_1)$. It follows that the A_y for all $\gamma_y \in F$ can be simultaneously isotoped (by small isotopies) so that their fibrations match afterwards. This yields a fibration of part of $\Sigma^o_{x;a,b}$ and, if $\Sigma_{x;a,b} \subset R_{\theta,\mu,-b^2}$, a global fibration.

If $e \in \Sigma_{x;a,b} - R_{\theta,\mu,-b^2}$ is a $(\theta,\mu,-b^2)$-edgy point, then the discussion in section 4.5.2 implies that $\partial B(e,\frac{1}{2}\theta^3\hat{s}_{\mu,-b^2}(e))$ lies in $R_{\theta,\mu,-b^2}$ and can be slightly isotoped inside $\Sigma_{x;a,b}$ to match the fibration obtained so far or, vice versa, the fibration can be adapted so that $\partial B(e,\frac{1}{2}\theta^3\hat{s}_{\mu,-b^2}(e))$ becomes a fiber. Let us call $\overline{B}(e,\frac{1}{2}\theta^3\hat{s}_{\mu,-b^2}(e))$ a *cap* of $\Sigma_{x;a,b}$. Since $\Sigma_{x;a,b}$ is compact and connected, it must have two disjoint caps which contain all $(\theta,\mu,-b^2)$-edgy points.

In the case when the neck cross section lies entirely in the 2-strained region, $\Sigma_{x;a,b} \subset R_{\theta,\mu,-b^2}$, we can sandwich it between two nearby neck cross sections and proceed as in section 4.5.2 (for S^2_κ) to isotope it by an isotopy supported nearby (i.e. in the sandwich) so that it becomes vertically saturated.

If the neck cross section $\Sigma_{x;a,b}$ contains edgy points and hence two caps, then we can co-

ordinate the fibration of $\Sigma_{x;a,b}$ with the fibration of the tubes T^j along the coarse edges, cf. section 4.5.2, so that the intersection $T^j \cap \Sigma_{x;a,b}$ consists of two fibers of T^j, namely one for each cap of $\Sigma_{x;a,b}$. (Here, we refer to the fibration of T^j by compact 2-orbifolds with one boundary component.) This is achieved by perturbing $\Sigma_{x;a,b}$ by a small isotopy (using the flow of the vector field L_{neck}) based near $T^j \cap \Sigma_{x;a,b}$ until it coincides with the closest tube cross sections of the T^j. Moreover, by a small isotopy of the fibration on $\Sigma_{x;a,b}$ minus the two caps, we can arrange that the intersections $S^j \cap \Sigma_{x;a,b}$ are fibers of the fibration of $\Sigma_{x;a,b}$. (In the latter step, we just use that any two noncontractible simple closed curves in an annular 2-orbifold are isotopic.)

4.5.4 Humps

We keep our assumption that (O, g) is $(v(\theta, \mu), -b^2)$-collapsed with $(v(\theta, \mu), s_0, K)$-curvature control below scale ρ_{-b^2} for sufficiently small $\theta, \mu > 0$. Then the discussion of sections 4.5.1, 4.5.2 and 4.5.3 applies.

Let $x \in O$ be a $(\theta, \mu, -b^2)$-hump as defined after 4.14. This means that on the scale $s(x) \in [s_1(\sigma, \mu)\rho_{-b^2}(x), \sigma\rho_{-b^2}(x)]$ of uniform conical approximation provided by 4.14, O is μ-well approximated in x by a flat disc of radius 1 with cone point of angle $\leq 2\pi - \theta$ or by a flat sector of radius 1 with angle $\leq \pi - \frac{\theta}{2}$. (This includes the half-open interval $[0, 1)$ as the degenerate case of the disc with cone angle 0.)

The closed ball $\overline{B}(x_i, \frac{1}{2}s(x_i))$ is a compact 3-suborbifold, since its boundary is almost orthogonal to radial (with respect to x) $\frac{\theta}{11}$-straight 1-strainers of length $\frac{1}{11}s(x)$, cf. 3.6, and hence a closed 2-suborbifold.

If $\mathrm{diam}(\partial B(x, \frac{1}{2}s(x)))$ is not too small, e.g. if $\mathrm{diam}(\partial B(x, \frac{1}{2}s(x))) > \theta^{\frac{49}{20}}\hat{s}_{\mu,-b^2}(x)$, then all points in $\partial B(x, \frac{1}{2}s(x))$ are edgy or 2-strained and, as above in section 4.5.3 for neck cross sections, we can construct a 1-dimensional fibration on most or all of $\partial B(x, \frac{1}{2}s(x))$. If the conical approximation in x is by a disc with a cone point, then $\partial B(x, \frac{1}{2}s(x)) \subset R^O_{\theta,\mu,-b^2}$ and the fibration is global. If the approximation is by a sector, then $\partial B(x, \frac{1}{2}s(x))$ contains edgy points close to the edges of the sector; these and the complement of the fibered region are covered by two caps whose boundaries are fibers. Since the edge cross sections ($\Sigma^o_{x;a,b}$, cf. section 4.5.2) associated to edgy points in $\partial B(x, \frac{1}{2}s(x))$ are also almost orthogonal to the radial direction (with respect to x), they can be embedded into $\partial B(x, \frac{1}{2}s(x))$ using an almost radial gradient like flow. As in section 4.5.3, we can coordinate the fibration of $\partial B(x, \frac{1}{2}s(x))$ with the fibration of the tubes T^j along the coarse edge so that the intersection $T^j \cap \partial B(x, \frac{1}{2}s(x))$ consists of two fibers of T^j, one for each cap of $\partial B(x, \frac{1}{2}s(x))$, and the intersections $\partial B(x, \frac{1}{2}s(x))$ are fibers of the fibration of $\partial B(x, \frac{1}{2}s(x))$. We say that the hump x has a *thick end*.

On the other hand, if $\mathrm{diam}(\partial B(x, \frac{1}{2}s(x)))$ is not too large, e.g. if $\mathrm{diam}(\partial B(x, \frac{1}{2}s(x))) < \theta_i^{\frac{401}{200}}\hat{s}_{\mu,-b^2}(x)$, then $\partial B(x_i, \frac{1}{2}s(x))$ is via an almost radial flow isotopic to a neck cross section associated to a $\frac{\theta}{11}$-straight 1-strainer of length $> \frac{1}{11}s(x) \geq \frac{1}{11}\hat{s}_{\mu,-b^2}(x)$ at a point in $\partial B(x, \frac{1}{2}s(x))$, and we have a *neck-hump* interface. Such an interface corresponds to a *thin* end of a neck (as defined in section 4.5.3).

We now have constructed a covering of the orbifold O by finitely many humps $B(x_i, \frac{1}{2}s(x_i))$, necks, tubes and the fibration of $R_{\theta,\mu,-b^2}$. This follows from Lemma 3.26: Consider a point

x which is not contained in the balls $B(x_i, \frac{1}{4}s(x_i))$ for the humps x_i; it admits an equilateral $< \theta$-straight 1-strainer with length $(\hat{s}_{\mu,-b^2}(x), \frac{3}{2}\hat{s}_{\mu,-b^2}(x))$. If the diameter of its cross section is $\geq \theta^{\frac{9}{4}}$ and $x \notin R_{\theta,\mu,-b^2}$, the lemma implies that x is contained in a tube with no end near x.

We now adjust the boundaries of humps, necks and tubes to our fibration of the 2-strained part $R_{\theta,\mu,-b^2}$. We have already done this in sections 4.5.2 and 4.5.3 for cyclic chain of edgy points and for thick ends of necks which contain no edgy points. We now proceed analogously for $\partial B(x_i, \frac{1}{2}s(x_i))$ for all $(\theta, \mu, -b^2)$-humps with $\operatorname{diam}(\partial B(x, \frac{1}{2}s(x_i))) > \theta^{\frac{49}{20}}\hat{s}_{\mu,-b^2}(x_i)$ and conical approximation by a disc with a cone point.

At this point, we discard all necks which are entirely contained in the union of humps, tubes and $R_{\theta,\mu,-b^2}$. By our previous discussion, every thick end of a hump or a neck meets $R_{\theta,\mu,-b^2}$. If they contain no edgy points, we have already isotoped them (by an isotopy supported nearby) to a vertically saturated 2-suborbifold.

Every thick end (of a hump or a neck) containing edgy points meets precisely two linear tubes along the coarse edge. By our discussion in section 4.5.2, these tubes can only end deep inside a neck or a hump, and hence a finite time after leaving the end intersect another hump or neck in a thick end. By the compactness of (O, g), such a sequence of tubes and thick ends must eventually close up. The union C of all humps, necks and tubes contained in such a closed chain is a topological 3-suborbifold with one boundary component $\partial_0 C$ a topological 2-suborbifold homeomorphic to T^2, K^2, Ann^2 or $M\ddot{o}b^2$, as follows from our discussion in sections 4.5.2 and 4.5.3. Of course, C may have other boundary components if it contains at least one neck.

For every chain C, we now extend the suborbifolds S_j^i (for the different tubes $T_j \subset C$) to closed 2-suborbifolds S_C^i isotopic (in C) to $\partial_0 C$, e.g. by forming suitable unions with the first three neck cross sections of a neck in C, and similarly for humps. The 2-suborbifold S_C^2 is then contained in $R_{\theta,\mu,-b^2}$, and we can apply our Waldhausen-like arguments from section 4.5.2 to isotope it (in the region between S_C^1 and S_C^3) so that it becomes vertically saturated with respect to the fibration of $R_{\theta,\mu,-b^2}$.

After performing this isotopy, we cut off the tubes T_j by a suitable cross section such that the fibrations on S_j^2 induced by T_j and $R_{\theta,\mu,-b^2}$ match. The complement of all humps, necks and tubes is now a saturated subset of the fibration of $R_{\theta,\mu,-b^2}$ (see remark after Definition 3.25).

We can cut off tubes by smooth cross sections transversal to the singular locus by using the uniqueness of differentiable structures (compare the discussion in section 4.5.2.) Then the components of our decomposition of a 3-orbifold (O, g) have piecewise smooth boundary and their interiors are disjoint open smooth 3-suborbifold.

Let us sum up our progress so far: For every $0 < \theta < \theta_1$ and $0 < \mu, < \mu_0(\theta)$ the following holds: If a 3-orbifold (O, g) is $(v(\theta, \mu), -b^2)$-collapsed with $(v(\theta, \mu), s_0, K)$-curvature control, it admits a decomposition (according to its coarse stratification) into topological 3-suborbifolds with disjoint interiors and piecewise smooth boundary, namely into $(\theta, \mu, -b^2)$-humps, necks, tubes and total spaces of orbifold fibrations with 1-dimensional fibers.

4.6 Local topology

In this section, we determine the topological structures of the components of the decomposition we constructed in the previous section. More precisely, we will determine the topology of cross sections to tubes and necks and the topological type of humps.

4.6.1 Tube and neck cross sections

In this section we prove that after decreasing θ further if necessary, we can control the topological type of the cross sections to tubes and necks in sufficiently collapsed 3-orbifolds.

The following proposition is related to an argument in the appendix of [FY92]; see also [MT08, 4.24] for a simplification of the special case needed here.

Proposition 4.28 (Topology of edge cross sections). *There exists $\theta_2 > 0$ such that for $\theta \in (0, \theta_2]$ and $\mu \in (0, \mu_0(\theta)]$ holds:*

If (O, g) is $(v(\theta, \mu), -b^2)$-collapsed with $(v(\theta, \mu), s_0, K)$-curvature control below scale ρ_{-b^2}, and if $x \in O$ is $(\theta, \mu, -b^2)$-edgy relative to a $< \theta$-straight 1-strainer (a, b) with length in $(\hat{s}_{\mu, -b^2}(x), \frac{3}{2}\hat{s}_{\mu, -b^2}(x))$, then the truncated cross section $\Sigma_{x;a,b} \cap \overline{B}(x, \frac{1}{2}\theta^3 \hat{s}_{\mu, -b^2}(x))$ is a connected compact 2-suborbifold with one boundary component and Euler characteristic $\chi \geq 0$.

Proof. Let $-b_i^2 \in [-1, 0)$ and let θ_i, μ_i be sequences of small positive numbers $\theta_i \to 0$ and $\mu_i \leq \mu_0(\theta)$. Suppose that the orbifolds (O_i, g_i) are $(v(\theta_i, \mu_i), -b_i^2)$-collapsed with $(v(\theta_i, \mu_i), s_0, K)$-curvature control below scale $\rho_{-b_i^2}$, and that the points $x_i \in O_i$ are $(\theta_i, \mu_i, -b_i^2)$-edgy relative to $< \theta_i$-straight 1-strainers (a_i, b_i) with lengths in $(\hat{s}_{\mu_i, -b_i^2}(x_i), \frac{3}{2}\hat{s}_{\mu_i, -b_i^2}(x_i))$.

We consider the neighborhoods of the points x_i on the scales $\theta_i^3 \hat{s}_{\mu_i, -b_i^2}(x_i)$. The rescaled pointed orbifolds $(\theta_i^3 \hat{s}_{\mu_i, -b_i^2}(x_i))^{-1} \cdot (O_i, x_i)$ Gromov-Hausdorff subconverge (collapse) to a 2-dimensional Alexandrov space with curvature ≥ 0 which splits off a line. In view of 3.20, this limit is the pointed flat halfplane with base point on the boundary. The cross sections $\Sigma_{x_i;a_i,b_i}$ converge to the cross sectional ray of the halfplane through the base point.

Let $z_i \in \Sigma_{x_i;a_i,b_i}$ with $d(x_i, z_i) = \theta_i^3 \hat{s}_{\mu_i, -b_i^2}(x_i)$. Then (a_i, b_i, x_i, z_i) is a $c\theta_i$-straight 2-strainer near the intersection $\gamma_i = \Sigma_{x_i;a_i,b_i} \cap \partial B(x_i, \frac{1}{2}\theta_i^3 \hat{s}_{\mu_i, -b_i^2}(x_i))$, for instance in the $\theta_i^4 \hat{s}_{\mu_i, -b_i^2}(x_i)$-neighborhood of it. Note that $(\theta_i^4 \hat{s}_{\mu_i, -b_i^2}(x_i))^{-1} \operatorname{diam}(\gamma_i) \to 0$, because $v(\theta_i, \mu_i) \leq (\theta_i^5 s_1(\sigma, \mu_i))^3$.

The following consideration applies for sufficiently large i. From the discussion in 4.4.3 we know that a neighborhood (of at least comparable size) of γ_i is fibered by the level sets of the $\mathbb{R}^2$-valued map $(f_{a_i,b_i}, d(x_i, \cdot))$. In particular, $\gamma_i \subset \Sigma_{x_i;a_i,b_i}$ is a connected closed 1-suborbifold. This fibration exists in fact on a larger region, for instance on a neighborhood of $\Sigma_{x_i;a_i,b_i} \cap \overline{A}(x_i, \frac{1}{100}\theta_i^3 \hat{s}_{\mu_i, -b_i^2}(x_i), \frac{99}{100}\theta_i^3 \hat{s}_{\mu_i, -b_i^2}(x_i))$. In particular, $d(x_i, \cdot)$ yields a product fibration (topologically) of $\Sigma_{x_i;a_i,b_i} \cap \overline{A}(x_i, \frac{1}{100}\theta_i^3 \hat{s}_{\mu_i, -b_i^2}(x_i), \frac{99}{100}\theta_i^3 \hat{s}_{\mu_i, -b_i^2}(x_i))$ over a compact interval. We note that as a consequence e.g. $\Sigma_{x_i;a_i,b_i} \cap B(x_i, \frac{3}{4}\theta_i^3 \hat{s}_{\mu_i, -b_i^2}(x_i))$ (deformation) retracts onto $\Sigma_{x_i;a_i,b_i} \cap \overline{B}(x_i, \frac{1}{2}\theta_i^3 \hat{s}_{\mu_i, -b_i^2}(x_i))$. Using the flow of a gradient like vector field X_i for the 1-strainer (a_i, b_i) as constructed in section 4.4.2, we obtain a homotopy of $B_i = \overline{B}(x_i, \frac{1}{2}\theta_i^3 \hat{s}_{\mu_i, -b_i^2}(x_i))$ into $\Sigma_{x_i;a_i,b_i} \cap B(x_i, \frac{3}{4}\theta_i^3 \hat{s}_{\mu_i, -b_i^2}(x_i))$ relative $\Sigma_i = \Sigma_{x_i;a_i,b_i} \cap B_i$, and together with the retraction $\Sigma_{x_i;a_i,b_i} \cap B(x_i, \frac{3}{4}\theta_i^3 \hat{s}_{\mu_i, -b_i^2}(x_i)) \to \Sigma_i$ a retraction $r_i : B_i \to \Sigma_i$. It is a retraction in the orbifold sense because X_i is tangential to the singular locus. We have $\partial \Sigma_i = \gamma_i$.

If B_i is homeomorphic to the closed 3-ball, then $\pi_1(\Sigma_i) \cong 1$ due to the retraction r_i, and hence Σ_i is a closed 2-disc. More generally, if B_i is discal then r_i lifts to an equivariant retraction $\tilde{r}_i : \tilde{B}_i \to \tilde{\Sigma}_i$ of manifold covers. As before, it follows that $\tilde{\Sigma}_i$ is a 2-disc and thus Σ_i is discal.

Suppose now that (after passing to a subsequence) none of the B_i is discal. We then determine the possible topological types of the B_i using the Shioya-Yamaguchi blow-up argument. Since the O_i also have $(v(\theta_i, \mu_i), s_0, K)$-curvature control below scale $\rho_{-b_i^2}$, according to our discussion in section 4.3.2, B_i is for large i homeomorphic to the product $[0,1] \times \Sigma_i'$ of the compact interval with a connected compact 2-orbifold with Euler characteristic $\chi \geq 0$ and one boundary component. Namely, after a suitable choice of base points, all blow-up limits are 3-dimensional of the form $(Y, y_0) = (\mathbb{R} \times W, (0, w_0))$ with a noncompact $\mathcal{C}^{10}$-smooth 2-orbifold W with sec ≥ 0, cf. 4.22 and 4.7. The topology of W is restricted by the (orbifold version of the) Soul Theorem. If $soul(W)$ is a point, then W is discal. If $\dim soul(W) = 1$, then W is a one-ended quotient of the flat cylinder $S^1 \times \mathbb{R}$, cf. 4.21. The relation between the topologies of Σ_i' (for a subsequence yielding the blow-up limit) and W is that W is homeomorphic to the interior of Σ_i'.

Knowing that $B_i \cong [0,1] \times \Sigma_i'$ for large i, we derive the topology of the truncated cross sections Σ_i using the embeddings $\Sigma_i \subset B_i \cong [0,1] \times \Sigma_i'$ and the retractions r_i as before. If Σ_i' is discal, then we saw above that also Σ_i is discal (and $\Sigma_i \cong \Sigma_i'$). Otherwise, Σ_i' is finitely covered by an annulus $\tilde{\Sigma}_i'$ and r_i lifts to an equivariant retraction $\tilde{r}_i : [0,1] \times \tilde{\Sigma}_i' \to \tilde{\Sigma}_i$ of smooth finite covers. Since the composition $\pi_1(\tilde{\Sigma}_i) \to \pi_1(\tilde{\Sigma}_i') \cong \mathbb{Z} \xrightarrow{(\tilde{r}_i)_*} \pi_1(\tilde{\Sigma}_i)$ of induced maps of fundamental groups is the identity, it follows that $\pi_1(\tilde{\Sigma}_i) \cong \mathbb{Z}$ or 0 and Σ_i is finitely covered by a 2-disc or an annulus. (We do not worry about excluding the case of the disc here.) $\qquad\square$

It follows from the proposition and our discussion in section 4.5.3 that for sufficiently small $\theta, \mu > 0$, *thick* ends of necks in $(v(\theta, \mu), -b^2)$-collapsed orbifolds with $(v(\theta, \mu), s_0, K)$-curvature control have Euler characteristic $\chi \geq 0$. Thus, we alredy control the topological structure of necks with at least one thick end. The following result generalizes this to arbitrary necks, i.e. cyclic necks or linear necks with two thin ends.

Proposition 4.29 (Topology of neck cross sections). *There exists $\theta_3 > 0$ such that for $\theta \in (0, \theta_3]$ and $\mu \in (0, \mu_0(\theta)]$ holds:*

If (O, g) is $(v(\theta, \mu), -b^2)$-collapsed with $(v(\theta, \mu), s_0, K)$-curvature control below scale ρ_{-b^2}, and if $x \in O$ is $(\theta, \mu, -b^2)$-necklike relative to a $< \theta$-straight 1-strainer (a, b) with length in $(\hat{s}_{\mu, -b^2}(x), \frac{3}{2}\hat{s}_{\mu, -b^2}(x))$, then the corresponding cross section $\Sigma_{x;a,b}$ is a closed 2-suborbifold with one boundary component and Euler characteristic $\chi \geq 0$.

Proof. Let $-b_i^2 \in [-1, 0)$ and let θ_i, μ_i be sequences of small positive numbers $\theta_i \to 0$ and $\mu_i \leq \mu_0(\theta)$. Suppose that the orbifolds (O_i, g_i) are $(v(\theta_i, \mu_i), -b_i^2)$-collapsed with $(v(\theta_i, \mu_i), s_0, K)$-curvature control below scale $\rho_{-b_i^2}$, and that the points $x_i \in O_i$ are $(\theta_i, \mu_i, -b_i^2)$-necklike relative to $< \theta_i$-straight 1-strainers (a_i, b_i) with lengths in $(\hat{s}_{\mu_i, -b_i^2}(x_i), \frac{3}{2}\hat{s}_{\mu_i, -b_i^2}(x_i))$.

We let $d_i < \theta_i^2$ denote the diameter of the cross sections $(\hat{s}_{\mu_i, -b_i^2}(x_i))^{-1} \cdot \Sigma_{x_i; a_i, b_i}$ and rescale by $2d_i^{-1}$. Then after passing to a subsequence, the orbifolds $(\frac{1}{2}d_i\hat{s}_{\mu_i, -b_i^2})^{-1} \cdot (O_i, x_i)$ converge to an Alexandrov space (Y, y) of curvature ≥ 0 which splits off a line. The factor of Y orthogonal to the line is the limit of the rescaled cross sections to the 1-strainers (a_i, b_i), and hence has

diameter 1 and is a compact Alexandrov space of curvature ≥ 0 and dimension 1 or 2.

If $\dim(Y) = 3$, by Lemma 4.7 Y is a $\mathcal{C}^{10}$-smooth orbifold and the convergence can be improved to $\mathcal{C}^5$-smooth. It follows that Y is isometric to $\Sigma \times \mathbb{R}$ for some closed 2-orbifold Σ of Euler characteristic $\chi(\Sigma) \geq 0$ and diameter 2. For sufficiently large i, we have an embedding of Σ into O_i which is transversal to the gradient-like vector field X_i for the strainer (a_i, b_i). This implies that Σ is isotopic to $\Sigma_{x_i;a_i,b_i}$.

If $\dim(Y) = 2$, Y must be isotopic to $[-1,1] \times \mathbb{R}$ or $S^1 \times \mathbb{R}$. Hence there are constants $\phi_i \to 0$ such that every point in $(\frac{1}{2} d_i \theta_i \hat{s}_{\mu_i, -b_i^2})^{-1} \cdot (O_i, x_i) \cap B(x_i, 100)$ either admits a $C_1 \phi_i$-straight 2-strainer of length ϕ_i^4 or lies within ϕ_i^3 of a point z admitting a $C_0 \phi_i$-straight 1-strainer (a', b') of length 1 such that $\Sigma_{z;a',b'}$ has diamter $\geq \phi_i^{\frac{5}{2}}$ and admits no $\frac{\pi}{2}$-straight 1-strainer of length ϕ_i^4.

If Y is isometric to $[-1,1] \times \mathbb{R}$, we construct two tubes of diameter $\approx \phi_i^3$ along $(\frac{1}{2} d_i \theta_i \hat{s}_{\mu_i, -b_i^2})^{-1} \cdot (O_i, x_i) \cap B(x_i, 100)$ as in section 4.5.2 corresponding to the two edges of Y. As described in section 4.5.3, we can now decompose $\Sigma_{x_i;a_i,b_i}$ into an annular part admitting a fibration by 1-dimensional orbifolds and two caps isotopic to cross sections of the two tubes. Note that for any $v > 0$ and sufficiently large i, the balls $(\frac{1}{2} d_i \theta_i \hat{s}_{\mu_i, -b_i^2})^{-1} \cdot (O_i, x_i) \cap B(z, 1)$ centered at points $z \in (\frac{1}{2} d_i \theta_i \hat{s}_{\mu_i, -b_i^2})^{-1} \cdot (O_i, x_i) \cap B(x_i, 10)$ uniformly have volume $< v$ with (v, s_0, K)-curvature control on scale 1 because of $1 << \rho_{-b_i^2}$. Thus we can proceed as in the proof of Proposition 4.28 to deduce that for sufficiently large i the cross sections to both tubes are compact 2-orbifolds with Euler characteristic $\chi \geq 0$ and one boundary component. This implies $\chi(\Sigma_{x_i;a_i,b_i}) \geq 0$.

If on the other hand Y is isometric to $S^1 \times \mathbb{R}$, it follows as in section 4.5.3 that for sufficiently large i the cross sections $\Sigma_{x_i;a_i,b_i}$ admit a global fibration by embedded 1-dimensional orbifolds and hence are toric (have Euler characteristic $\chi = 0$). $\qquad\square$

4.6.2 Humps

For the remaining part of the proof, we fix $\bar{\theta} > 0$ such that the results of section 4.5 and Propositions 4.28 and 4.29 apply. Thus whenever $\mu < \mu_0(\bar{\theta})$ and a 3-orbifold (O, g) is $(v(\bar{\theta}, \mu), -b^2)$-collapsed with $(v(\bar{\theta}, \mu), s_0, K)$-curvature control, it admits a decomposition according to its coarse stratification and we have control over the cross sections of all tubes and necks.

In order to determine the local topology of humps we will improve the quality of our conical approximations, i.e. make μ sufficiently small. Again, we will adjust the upper bound for μ in several steps.

We say that a $(\bar{\theta}, \mu, -b^2)$-hump $x \in (O, g)$ is a *thick hump* if O can in x be μ-well approximated on scale $s(x)$ by a flat cone with a base of diameter $\in (\frac{\pi}{4}, \pi - \frac{\bar{\theta}}{2})$. In particular, this excludes the case of conical approximation by the 1-dimensional cones $(-1, 1)$ or $[0, -1)$. A thick hump must have a thick end in the sense of section 4.5.4.

Proposition 4.30 (Topological type of thick humps). *There exists $0 < \mu_1 < \mu_0(\bar{\theta})$ such that:*

Let $0 < \mu < \mu_1$. If (O, g) is $(v(\bar{\theta}, \mu), -b^2)$-collapsed with $(v(\bar{\theta}, \mu), s_0, K)$-curvature control and if $x \in O$ is a thick $(\bar{\theta}, \mu, -b^2)$-hump, then $B(x, \frac{1}{2} s(x))$ is discal or solid toric.

Proof. This is an application of the Shioya-Yamaguchi blow-up discussed in section 4.3.2. Let

$b_i^2 \in [-1, 0)$ and $\mu_i \to 0$. Suppose that the orbifolds (O_i, g_i) are $(v(\bar{\theta}, \mu_i), -b_i^2)$-collapsed with $(v(\bar{\theta}, \mu_i), s_0, K)$-curvature control and that the points $x_i \in O_i$ are thick $(\bar{\theta}, \mu_i, -b_i^2)$-humps, i.e. μ_i-well approximated on scale $s(x_i)$ by flat cones C_i with bases of diameter $\in (\frac{\pi}{4}, \pi - \frac{\bar{\theta}}{2})$.

A subsequence of the cones C_i converges to a flat cone C_∞ with a base of diameter $\in [\frac{\pi}{4}, \pi - \frac{\bar{\theta}}{2}]$. It follows that a subsequence of the rescaled balls $s(x_i)^{-1} \cdot B(x_i, s(x_i))$ also converges to C_∞ in the Gromov-Hausdorff sense.

Unless infinitely many of the balls $B(x_i, s(x_i))$ are discal, by our results in section 4.3.2 there is a sequence of rescaling factors $\delta_i \to 0$ such that the sequence $(\delta_i s(x_i))^{-1} \cdot B(x_i, s(x_i))$ Gromov-Hausdorff subconverges to a 3-dimensional limit space (Y, y) of curvature ≥ 0. As discussed in 4.3.1, (Y, y) is actually a $\mathcal{C}^{10}$-smooth 3-orbifold and the convergence can be improved to $\mathcal{C}^5$-smooth.

The soul of the blow-up limit Y must be a point or 1-dimensional since it cannot be 2-dimensional by 4.3.2. Hence Y must be either discal or solid toric. Again by 4.3.1, this implies that for sufficiently large i, the balls $B(x_i, \frac{1}{2}s(x_i))$ are also either discal or solid toric. $\qquad \square$

We can "read off" the topological type of a thick hump from the components of the decomposition it intersects. Let $\mu > 0$ be sufficiently small and suppose that the orbifold (O, g) is sufficiently volume collapsed with curvature control such that it admits a decomposition according to its coarse stratification and that Proposition 4.30 holds. Let $x \in O$ be a thick hump of this decomposition.

If O is in x μ-well approximated on scale $s(x)$ by a flat cone over a circle, equivalently if $\partial B(x, \frac{1}{2}s(x))$ is contained in $R_{\bar{\theta}, \mu, -b^2}$, it follows that $\partial B(x, \frac{1}{2}s(x))$ admits a fibration by 1-dimensional fibers and hence cannot be spherical. This implies that the hump x is a solid toric 3-suborbifold bounded by a vertically saturated component of a 1-fibered component of the decoposition of O.

If the conical approximation of O in x is by a flat sector, $\partial B(x, \frac{1}{2}s(x))$ intersects precisely two tubes with cross sections Σ_1, Σ_2. Both cross sections have Euler characteristic $\chi \geq 0$. Since $\partial B(x, \frac{1}{2}s(x))$ can be decomposed into a union of Σ_1, Σ_2 and an annular (1-fibered) component, we have $\chi(\partial B(x, \frac{1}{2}s(x)) = \chi(\Sigma_1) + \chi(\Sigma_2)$. Thus if $\chi(\Sigma_1) = \chi(\Sigma_2) = 0$, it follows that $\chi(\partial B(x, \frac{1}{2}s(x)) = 0$ and hence that the hump x is again solid toric. Conversely, if at least one of the Σ_i is discal, the hump x must be discal as well.

We cannot expect that the arguments from the proof of the last Proposition also work for not necessarily thick humps, i.e. humps with conical approximation by cones with a base of arbitrarily small diameter.

In this case, it is possible that the $s(x_i)^{-1} \cdot B(x_i, s(x_i))$ collapse to a 1-dimensional cone and that the rescaled blow-ups $(\delta_i s(x_i))^{-1} \cdot B(x_i, s(x_i))$ only Gromov-Hausdorff converge to a 2-dimensional Alexandrov space (Y, y) of non-negative curvature (see section 4.3.2). We will however see that in this case we can again apply our arguments from section 4.5 to obtain a decomposition of the humps $B(x_i, s(x_i))$ with respect to the scale $\delta_i s(x_i)$ such that no thin humps or necks occur in the decomposition. When investigating collapse to the 2-dimensional space Y, we operate on the scale $\delta_i s(x_i)$ rather than on the natural curvature scale $\rho_{-b_i^2}$. Equivalently, the rescaled orbifolds $(\delta_i s(x_i))^{-1} \cdot (O_i, x_i)$ collapse to Y on scale 1. Note that we have already encountered a similar situation (in a very restricted setting) in the proof of

Proposition 4.29.

Throughout the following considerations, we will always assume that $\rho_{-b^2} \gg 1$. This implies $\sec \geq -b^2 \geq -1$ on balls of radius 1. Moreover, it means that (v, s_0, K)-curvature control on scale ρ_{-b^2} implies (v, s_0, K)-curvature control on scale 1.

We ommit $-b^2$ in our notation to indicate that we work on scale 1 rather than ρ_{-b^2}. For instance, we say that a 3-orbifold is v-collapsed at a point p if $\operatorname{vol} B(p, 1) < v$. Similarly, for $\theta, \mu > 0$ we define the set $R_{\theta,\mu}$ as the set of all points admitting $< C_1\theta$-straight 2-strainers of length $> \theta^4 s_1(\sigma, \mu)$. Similarly, we define (θ, μ)-edgy points, (θ, μ)-necklike points and (θ, μ)-humps.

We can adapt our previous results to our new setting of collapse at scale 1. More precisely, we have

Lemma 4.31. *There are $\hat{\theta} > 0$ and $0 < \hat{\mu} < \mu_0(\hat{\theta})$ such that:*

Let (O, g) be a 3-orbifold with $\sec \not\geq 0$ and $x \in O$. Suppose that for some $R > 0$ the orbifold O is on the ball $B(x, 6R)$ $(v(\hat{\theta}, \hat{\mu}))$-collapsed with $(v(\hat{\theta}, \hat{\mu}), s_0, K)$-curvature control on scale $1 \ll \rho_{-b^2}$, and that $\operatorname{diam} O \geq 6R$. Then the following hold:

- *The orbifold O can in every $y \in B(x, 5R)$ be $\hat{\mu}$-well approximated on some scale $s(y) \in [s_1(\sigma, \hat{\mu}), \sigma]$ by a cone of dimension 1 or 2.*

- *There exists an open subset U, $R_{\hat{\theta},\hat{\mu}} \cap B(x, 3R) \subseteq U \subset B(x, 4R)$ such that every connected component of U is the total space of a smooth orbifold fibration with fiber S^1 or the mirrored interval $\bar{I}^1$, and all fibers have angle $< \nu$ with the almost vertical line field $L_{\hat{\theta},\hat{\mu}}$.*

- *If $y \in B(x, 4R)$ is $(\hat{\theta}, \hat{\mu})$-edgy relative to an equilateral $\hat{\theta}$-straight 1-strainer (a, b) with length in $(s_1(\sigma, \hat{\mu}), \frac{3}{2} s_1(\sigma, \hat{\mu}))$, then the truncated cross section $\Sigma_{y;a,b} \cap \overline{B}(y, \frac{1}{2}\hat{\theta}^3 s_1(\sigma, \hat{\mu}))$ is a connected compact 2-suborbifold with one boundary component and Euler characteristic $\chi \geq 0$.*

- *If $y \in B(x, 4R)$ is a thick $(\hat{\theta}, \hat{\mu})$-hump, then $B(y, \frac{1}{2}s(y))$ is discal or solid toric.*

Proof. The proof works exactly as for Propositions 4.14, 4.24, 4.28 and 4.30 since in all of these proofs we rescale by the collapse scale anyway. $\square$

We now return to our original discussion of the topological structure of general humps in a $(v(\bar{\theta}, \mu), -b^2)$-collapsed 3-orbifold with $(v(\bar{\theta}, \mu), s_0, K)$-curvature control.

Proposition 4.32. *There exists $0 < \mu_2 < \mu_0(\bar{\theta})$ such that:*

Let $0 < \mu < \mu_2$. If (O, g) is $(v(\bar{\theta}, \mu), -b^2)$-collapsed with $(v(\bar{\theta}, \mu), s_0, K)$-curvature control and if $x \in O$ is any $(\bar{\theta}, \mu, -b^2)$-hump, then one of the following holds:

1. *$B(x, \frac{1}{2}s(x))$ is discal or solid toric.*

2. *$B(x, \frac{1}{2}s(x))$ is has the topological type of $(\Sigma \times [-1, 1])/\mathbb{Z}_2$ with Σ a closed 2-orbifold with $\chi(\Sigma) \geq 0$ and $\mathbb{Z}_2$ operating as a reflection on $[-1, 1]$.*

3. $B(x, \frac{1}{2}s(x))$ admits a decomposition as in section 4.5 into a 1-fibered part, tubes, humps and precisely one neck containing $A(x, \frac{1}{4}s(x), \frac{1}{2}s(x))$. The cross section of this neck is a closed 2-orbifold with $\chi \geq 0$. Finally, all the humps occuring in this decomposition are discal or solid toric.

Proof. Again, let $b_i^2 \in [-1, 0)$ and $\mu_i \to 0$. Suppose that the orbifolds (O_i, g_i) are $(v(\bar{\theta}, \mu_i), -b_i^2)$-collapsed with $(v(\bar{\theta}, \mu_i), s_0, K)$-curvature control and that the points $x_i \in O_i$ are any $(\bar{\theta}, \mu_i, -b_i^2)$-humps, i.e. μ_i-well approximated on scale $s(x_i)$ by flat cones C_i with bases of diameter $< \pi - \frac{\bar{\theta}}{2}$.

A subsequence of the cones C_i converges to some flat cone C_∞ with a base of diameter $< \pi - \frac{\bar{\theta}}{2}$, and thus a subsequence of the rescaled balls $s(x_i)^{-1} \cdot B(x_i, s(x_i))$ also converges to C_∞ in the Gromov-Hausdorff sense.

If the cone C_∞ is 2-dimensional, the proof proceeds as for Proposition 4.30 to show that for sufficiently large i we are in the first case of the proposition.

We now suppose that C_∞ is 1-dimensional. It is then isometric to the half-open interval $[0, 1)$ with cone point $\{0\}$. Moreover we suppose that for infinitely many i the ball $B(x_i, s(x_i))$ is not discal.

By our discussion in 4.3.2 we therefore have rescaling factors $\delta_i \to 0$ such that the sequence $(\delta_i s(x_i))^{-1} \cdot B(x_i, s(x_i))$ Gromov-Hausdorff subconverges to a non-compact limit Alexandrov space (Y, y) of curvature ≥ 0 and dimension 2 or 3.

In case $\dim(Y) = 3$ Y is again a $\mathcal{C}^{10}$-smooth orbifold and we can improve the convergence to $\mathcal{C}^5$-smooth. Depending on the dimension of its soul, Y is discal, solid toric or diffeomorphic to $(\Sigma \times [-1, 1])/\mathbb{Z}_2$ with $\chi(\Sigma) \geq 0$ and $\mathbb{Z}_2$ operating on $[-1, 1]$ by a reflection. (We can exclude a product structure since the $B(x_i, \frac{1}{2}s(x_i))$ and hence Y are one-ended.) To finish this case, we note again that $B(x_i, \frac{1}{2}s(x_i))$ is homeomorphic to Y for sufficiently large i by our discussion in 4.3.1.

We are now left with the case where C_∞ is isometric to $[0, 1)$ and the pointed blow-up limit (Y, y) is 2-dimensional. Remember from 4.3.1 that we have a concave 1-Lipschitz function β_ξ on Y coming from the unique direction at $\{0\} \in [0, 1)$. By construction, y is a maximum of β with $\beta(y) = 0$.

We observe two important properties of the space Y with respect to its curvature bound ≥ 0:

(i) For every point $z \in Y$, there is a $\frac{\pi}{2}$-straight 1-strainer of length $\frac{1}{2}$ centered at z.

More precisely, let ρ_ξ^z be a ray of maximal β_ξ-decay emanating from z and let $y' \in Y$ be a maximum of β_ξ, i.e. $\beta_\xi(y') = 0$. Recall that by construction, there is a critical point x at distance 1 from y with $\beta_\xi(x) = 0$. Concavity of β_ξ implies $\beta_\xi = 0$ on the whole segment yx of length 1. Hence we can choose y' such that $d(z, y') \geq \frac{1}{2}$. Since β_ξ is 1-Lipschitz, we have $d(\rho_\xi^z(t), y') \geq |\beta_\xi(\rho_\xi^z(t))| \geq t$ which implies that for arbitrarily small $\epsilon > 0$ and sufficiently large $t > t_0(\epsilon)$ we have $\angle_z(y', \rho_\xi^z(t)) \geq \frac{\pi}{2} - \epsilon$. This implies property (i).

(ii) There is a radius R (depending on $\bar{\theta}$ and Y) such that for every point $x \in Y$ with $d(z, y) = r \geq R$ there is a $\bar{\theta}$-straight 1-strainer zxz' of length r.

Otherwise, we could find a sequence of points $z_i \to \infty$ with $d(z_{i+1}, y) \geq 2d(z_i, y)$ such

that $\tilde{\angle}_{z_i}(y, z_j) \leq \pi - \frac{\bar{\theta}}{4}$ for $i < j$. After passing to a subsequence, we can assume that moreover $\tilde{\angle}_{z_j}(y, z_i) \leq \frac{\bar{\theta}}{8}$: We only have to make sure that $d(y, z_i)$ is growing sufficiently fast, i.e. $d(y, z_{i+1}) \geq \lambda d(y, z_i)$ for some $\lambda(\bar{\theta})$. But this implies that $\angle_y(x_i, x_j) \geq \tilde{\angle}_y(x_i, x_j) \geq \frac{\bar{\theta}}{4}$ for all $i \neq j$ which is absurd.

Since we have by construction $(\delta_i s(x_i))^{-1} \cdot B(x_i, s(x_i)) \to (Y, y)$, for sufficiently large i the $6R$-balls in $(\delta_i s(x_i))^{-1} \cdot B(x_i, s(x_i))$ are $v(\hat{\theta}, \hat{\mu})$-collapsed with $(v(\hat{\theta}, \hat{\mu}), s_0, K)$-curvature control on scale $1 << \rho_{-b^2}$. Thus Lemma 4.31 applies to these balls.

In particular, for these i every point $z \in B_{(\delta_i s(x_i))^{-1}g_i}(x_i, 6R)$ can be $\hat{\mu}$-well be approximated on scale $s(z) \in [s_1(\sigma, \hat{\mu}), \sigma]$ by a flat cone.

If we make i sufficiently large (so that $d_{GH}(B_{(\delta_i s(x_i))^{-1}g_i}(x_i, 6R), B(y, 6R))$ becomes sufficiently small), we also have conical approximation on the ball $B(y, 6R) \in Y$ of slightly lower quality, say $2\hat{\mu}$-good approximation. On the other hand, it follows from properties (i) and (ii) and the fact that Y contains a flat strip of width 1 (see 4.3.2) that the diameters of approximating cones must be $> \frac{\pi}{4}$ on $B(y, 6R)$ and $> \pi - \frac{\hat{\theta}}{2}$ on $A(y, R, 6R)$. Hence for i sufficiently large we can deduce that the same bounds on the diameters of approximating cones hold on the balls $B_{(\delta_i s(x_i))^{-1}g_i}(x_i, 6R)$.

We can now apply Proposition 3.7 to obtain a decomposition of $B_{(\delta_i s(x_i))^{-1}g_i}(x_i, 6R)$ into finitely many $(\hat{\theta}, \hat{\mu})$-humps and the set $S^i_{\hat{\theta}, \hat{\mu}}$ of points admitting $\hat{\theta}$-straight 1-strainers with length in $(\frac{1}{11}s_1(\sigma, \hat{\mu}), \frac{3}{22}\frac{1}{11}s_1(\sigma, \hat{\mu}))$. Note that all humps must lie in $B_{(\delta_i s(x_i))^{-1}g_i}(x_i, R)$ and are *thick*.

We now proceed as in Lemma 4.31 and construct a covering of $B_{(\delta_i s(x_i))^{-1}g_i}(x_i, 3R)$ by the total spaces of fibrations with 1-dimensional fibers, tubes and thick humps as in 4.5. Note that all occuring humps are discal or solid toric.

By construction, the region $N = A_{(\delta_i s(x_i))^{-1}g_i}(x_i, 2R, 6R)$ is homeomorphic to a product of the interval and $\partial B(x_i, s(x_i))$. We add it as a "neck" to our covering of $B_{(\delta_i s(x_i))^{-1}g_i}(x_i, 3R)$ and note that all points on its inner boundary $\partial B_{(\delta_i s(x_i))^{-1}g_i}(x_i, 2R)$ are edgy or 2-strained. Moreover, property (ii) of (Y, y) implies that we can find $< \theta$-straight 1-strainers of length in $(\frac{1}{11}s_1(\sigma, \hat{\mu}), \frac{3}{22}s_1(\sigma, \hat{\mu}))$ which are almost orthogonal to $\partial B_{(\delta_i s(x_i))^{-1}g_i}(x_i, 2R)$ at every point $z \in \partial B_{(\delta_i s(x_i))^{-1}g_i}(x_i, 2R)$ As in 4.5.4, this allows us to construct a fibration on most or all of $\partial B_{(\delta_i s(x_i))^{-1}g_i}(x_i, 2R)$, which is either global or capped off by the cross sections of two tubes. In other words, we treat $\partial B_{(\delta_i s(x_i))^{-1}g_i}(x_i, 2R)$ as a thick end of the "neck" N.

We now adjust the interfaces in $B_{(\delta_i s(x_i))^{-1}g_i}(x_i, 3R)$ as in 4.5.4 and extend N radially away from x_i to $\partial B(x_i, \frac{1}{2}s(x_i))$ using an gradient-like vector field for $d(x_i, .)$. (Recall that there are no critical points for x_i at distance $> \delta_i s(x_i)$.) Thus for sufficiently large i we have constructed a decomposition as in the third case of the proposition. $\square$

4.6.3   Proof of the main result

In this section we complete the proof of Theorem 4.11. As explained in section 4.1, Theorem 4.11 together with Corollary 2.9 implies our main result Theorem 4.13.

In addition to $\bar{\theta} > 0$ from section 4.6.2, we fix some $0 < \bar{\mu} \leq \mu_2$ with μ_2 as in Proposition 4.32. We set $v = v(\bar{\theta}, \bar{\mu})$. Then our discussion so far implies the following:

Let (O, g) be a closed connected 3-orbifold with sec $\not\geq 0$ and $\mathrm{rad}(O) \geq \frac{1}{2}\rho_{-b^2}$ for some $-b^2 \in [-1, 0)$ and which contains no bad 2-suborbifold. Suppose that (O, g) is $(v, -b^2)$-collapsed with (v, s_0, K)-curvature control on scale ρ_{-b^2}. Then O admits a decomposition according to its coarse stratification into finitely many components of the following kind: total spaces of orbifold fibrations with 1-dimensional fibers, tubes and necks with cross sections of Euler characteristic $\chi \geq 0$, and humps which are solid toric, 3-discal or homeomorphic to $(\Sigma \times [-1, 1])/\mathbb{Z}_2$ as in Proposition 4.32.

There are three possibilities for every end of a neck or hump: If the end is thin, a hump ends in a neck and vice versa. If the end is thick and meets no tubes, it intersects one of the components with 1-dimensional fibration. In this case the end is toric and vertically saturated with respect to this fibration. Finally, there is the possibility that a thick end meets precisely two tubes and one of the components with 1-dimensional fibration. The boundary component of such an end can then be further decomposed into cross sections of the two tubes and an annular part between them which again is vertically saturated with respect to the 1-dimensional fibration. The end can be spherical or toric depending on the Euler characteristic of the tube cross sections.

In order to complete the proof of Theorem 4.11 it therefore suffices to show that after performing a finite number of surgeries such a decomposition can be simplified to a graph decomposition.

Because O admits no bad 2-suborbifolds, the cross sections of all necks must be spherical or toric. We perform surgery across all necks with spherical cross section. If such a neck bounds a discal component or one of type $(\Sigma \times [-1, 1])/\mathbb{Z}_2$ for some closed spherical 2-orbifold Σ (corresponding to a hump at a thin end of the neck) the resulting summmand is a finite quotient of S^3. Similarly, if O is a cyclic neck with spherical cross section, it is decomposed by one surgery into a finite quotient of the 3-sphere. We also perform surgery along the boundaries of all components of type $(\Sigma \times [-1, 1])/\mathbb{Z}_2$ with spherical Σ which are not adjacent to a neck, i.e. coming from humps with thick ends.

The orbifold O may now be disconnected. We discard all spherical summands. Every remaining summand admits a decomposition as above without necks with spherical cross sections or humps of type $(\Sigma \times [-1, 1])/\mathbb{Z}_2$ for spherical Σ.

Let V be a 3-discal component of this decomposition. It meets two coarse edges of O; let T and T' be the corresponding tubes. (We do not exclude the case $T = T'$.) Due to Euler characteristic reasons, at least one of the tube cross sections Σ_T, $\Sigma_{T'}$ must be discal. If both are discal, then the two cross sections must be homeomorphic because otherwise ∂V would be a bad 2-suborbifold of O. In this case, V is homeomorphic to $\Sigma_T \times [0, 1]$ and we can replace the union $T \cup V \cup T'$ by a single tube with cross section Σ_T, thereby simplifying the decomposition.

Because of the finiteness of the decomposition of O, after repeating this step a finite number of times we can assume that no 3-discal component of the decomposition of O meets two tubes with discal cross section.

Consider now a tube T with discal cross section. If T is cyclic, it is homeomorphic to a fibration over S^1 with discal fiber and hence to a solid toric 3-orbifold with boundary. If T is linear, it ends in two 3-discal components V_1 and V_2 such that the other tubes ending in the

V_i have *annular* cross section. In this case, the union $V_1 \cup T \cup V_2$ is again solid toric, cf. the discussion after Proposition 2.2. By considering these solid toric suborbifolds as components of our decomposition, we therefore can assume that all tubes occuring in the decomposition of O have annular cross section.

We recall from sections 4.5.2 and 4.5.4 that for each remaining tube T (which now must have annular cross section) intersecting the total space U of a fibration with 1-dimensional fiber, the two fibrations of U and T match on the 2-suborbifold $T \cap U$. Since every annular 2-orbifold inherits an orbifold fibration with 1-dimensional fiber from the fibration of the annulus by circles, we can extend the fibration of U to a Seifert fibration of $T \cup U$.

We have now obtained a decomposition along disjoint embedded toric 2-suborbifolds into components which are total spaces of orbifold Seifert fibrations, solid toric suborbifolds, necks with toric cross section and components of type $(\Sigma \times [-1,1])/\mathbb{Z}_2$ with toric Σ. This is a graph decomposition by definition (cf. section 2.3.3). The proof of Theorem 4.11 is now complete. $\square$

5 An extension to the case with boundary

In this section we extend the results of the previous one to a somewhat larger class of volume collapsed 3-orbifolds.

We define a *hyperbolic orbifold cusp* to be a complete 3-orbifold with boundary which is isometric to the quotient of a horoball in hyperbolic 3-space by a cocompact isometric group action. Thus, a hyperbolic orbifold cusp is diffeomorphic to $\Sigma^2 \times [0, \infty)$ for some toric orbifold Σ^2 (by Bieberbach's theroem). With the construction of the Ricci flow with surgery in mind (cf. [Pe03] and [KL10] for orientable manifolds), we will consider hyperbolic orbifold cusps with sectional curvature equal to $-\frac{1}{4}$.

Definition 5.1 (Almost cuspidal ends). A Riemannian 3-orbifold (O, g) with boundary has (v, s_0)-almost cuspidal ends if for every component $C \subset \partial O$ there is a hyperbolic orbifold cusp X_C such that the pairs $(N_{100}(C), C)$ and $(N_{100}(\partial X_C), \partial X_C)$ habe distance $\leq v$ in the $\mathcal{C}^{s_0}$-topology.

The following theorem generalizes Theorem 4.13 to locally volume collapsed 3-orbifolds with almost cuspidal ends (compare again [Pe03, Theorem 7.4], [MT08, Theorem 0.2] and [KL10, Theorem 1.3]).

Theorem 5.2. *Let $s_0 \in \mathbb{N}$ and let $K : (0, \omega_3) \to (0, \infty)$ be a function. If s_0 is sufficiently large, then there exists a constant $v_0 = v_0(s_0, K) \in (0, \omega_3)$ such that:*

If (O, g) is closed or compact with (v_0, s_0)-almost cuspidal ends, is $(v_0, -1)$-collapsed, has (v_0, s_0, K)-curvature control below the scale ρ_{-1} and contains no bad 2-suborbifolds, then O is either closed and admits a metric with $\sec \geq 0$, or satisfies Thurston's Geometrization Conjecture.

Proof. Throughout the following proof, we choose s_0, $\bar{\theta}$, $\bar{\mu}$ and $v = v(\bar{\theta}, \bar{\mu})$ as in the proof of Theorem 4.11.

If (O, g) is closed, $(v, -1)$-collapsed, has (v, s_0, K)-curvature control below scale ρ_{-1} and contains no bad 2-suborbifolds, we have already shown that the theorem holds.

We therefore now suppose that (O, g) has at least one $((v, s_0)$-cuspidal) end. In this case, we first observe that $\rho_{-1}(x) \approx 4$ near a boundary component C, say on $A(C, 10, 90)$. This means that there are points $x \in O$ with $\operatorname{diam} O >> 2\rho_{-1}(x)$. Hence collapse to a point cannot occur and we can work with $-b^2 = -1$. (In other words, we do not need to make use of the more general setting of Theorem 4.11.)

After decreasing v if necessary, we obtain that every point x close to a cuspidal end (again, say on $A(C, 10, 90)$ for a boundary component $C \subset \partial O$) admits a $< \bar\theta$-straight 1-strainer of length $\hat{s}_{\bar\mu, -1}(x)$ almost orthogonal to level sets of $d(C, .)$. In particular, we conclude $A(C, 10, 90) \subset S_{\bar\theta, \bar\mu, -1}$.

We fix this new value of v and suppose from now on that (O, g) is compact with (v, s_0)-almost cuspidal ends and $(v, -1)$-collapsed, has (v, s_0, K)-curvature control below scale ρ_{-1} and contains no bad 2-suborbifolds.

We now define the *cusped necks* of O to be the closed sets $\overline{N}_{25}(C)$ for all boundary components $C \subset \partial O$. On the neighbourhoods $N_{90}(C)$ of the cusped necks, we have smooth gradient-like vector fields V_C for the distance function $d(C, .)$. Cusped necks are homeomorphic to $\Sigma^2 \times [0, 1]$ for some toric 2-orbifold Σ^2. Throughout the following discussion, we are only interested in the ends of cusped necks which are not boundary components of O.

By construction, cusped necks are disjoint from each other; they are also disjoint from the humps in $O \setminus \bigcup_C N_{10}(C)$ by 3.6 (i).

We will now show how cusped necks can be integrated in our decomposition of O according to its coarse stratification much like humps. As with humps, we call the end of a cusped neck *thin* if the diameter of $\{d(C, .) = 25\}$ is not too large, say $\leq \bar\theta^{\frac{401}{200}} s_1(\bar\mu, -1)$. (Remember that we have seen $\rho_{-1} \approx 4$ on a large neighbourhood of the end, so the above condition means that the diameter is of order $\theta^{\frac{401}{200}} \hat{s}_{\bar\mu, -1}$ for all points in this neighbourhood.) A thin end of a cusped neck corresponds to the *thin end* of a neck (as defined in 4.5.3) and the interface can be matched up using he flow of V_C.

Similarly, we say that the end of a cusped neck is *thick* if the diameter of $\{d(C, .) = 25\}$ is sufficiently large, say $\geq \bar\theta^{\frac{49}{20}}$. In this case, we can proceed as we did for humps in section 4.5.4 and construct a 1-dimensional fibration on all or almost all of $\{d(C, .) = 25\}$, with the possible exception of two tube cross sections. We also can perturb $\{d(C, .) = 25\}$ such that it intersects the tubes (if there are any) in tube cross sections (again, using the flow of V_C). If the end of a cusped neck intersects two tubes, it follows immediately that both tubes have *annular* cross sections.

We now proceed to construct a decomposition of O as we did in the closed case. After adjusting the interfaces of the different components of the decomposition (using our Waldhausen-type arguments) and performing a finite number of surgeries we again obtain components which are spherical or admit a further decomposition along (piecewise smooth) toric suborbifolds into pieces which are orbifold Seifert fibrations, solid toric suborbifolds, necks with toric cross section or components of type $(\Sigma \times [-1, 1])/\mathbb{Z}_2$ with toric Σ. (The new components coming from cusped necks of O are topologically of the same kind as necks with toric cross sections.) These

decompositions are again graph, which by virtue of Corollary 2.9 completes the proof of the theorem. □

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